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Abstract

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MATHEMATICS

M. V. FEDORYUK

ASYMPTOTICS OF THE DISCRETE SPECTRUM OF THE OPERATOR $-w''(x) + \lambda^2 p(x) w(x)$

(Presented by Academician A. A. Dorodnitsyn, 17 IV 1964)

Consider the eigenvalue problem

$$w''(x) - \lambda^2 p(x) w(x) = 0, \qquad w(\pm\infty) = 0, \tag{1}$$

where $p(z)$ is an entire function of z. We study the asymptotics of the discrete spectrum of problem (1). Our method is as follows. Take a solution $w(x,\lambda)$ such that $w(+\infty,\lambda) = 0$, continue its asymptotics into the complex z-plane, find the asymptotics of $w(x,\lambda)$ as $x \to -\infty$, and determine the eigenvalues from the equation $w(-\infty,\lambda) = 0$.

1. We first formulate all results for polynomials $p(z)$. Denote

$$\alpha_0(z) = -\frac{p'(z)}{4p(z)}, \qquad \alpha_{k+1}(z) = -\frac{1}{2\sqrt{p(z)}}\left(\sum_{m=0}^{k} \alpha_m(z)\alpha_{k-m}(z) + \alpha_k'(z)\right). \tag{2}$$

In Theorems 1-3 it is assumed that $p(z)$ is a polynomial with real coefficients, all real zeros of $p(z)$ are simple, and $p(\pm\infty) = +\infty$.

Theorem 1. *Suppose that $p(x)$ has exactly two zeros $x_1 < x_2$. Then as $n \to \infty$ there is the asymptotic expansion*

$$\lambda_n \int_C \sqrt{p(z)}\, dz \sim 2\pi n i + \pi i + \sum_{k=1}^{\infty} \lambda_n^{-k} \int_C \alpha_k(z)\, dz. \tag{3}$$

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Here $0 < \lambda_1 < \lambda_2 < \cdots$ are the eigenvalues of problem (1), C is a contour in the z-plane enclosing the segment $[x_1, x_2]$ and containing no other zeros of $p(z)$; $\sqrt{p(x)} > 0$ for $x \in C,\ x > x_2$. The coefficient of λ_n is equal to

$$2i \int_{x_1}^{x_2} \sqrt{|p(x)|}\, dx.$$

From formula (3) it is not difficult to obtain an asymptotic series in powers of $1/n$ for λ_n.

Theorem 2. *Suppose that $p(x)$ has $2k$ zeros x_j, $x_j < x_{j+1}$. Then for large n the spectrum of problem (1) consists of k series of eigenvalues λ_{nj}, and as $n \to \infty$ for λ_{nj} formula (3) holds with C replaced by C_j, where C_j is a contour enclosing the segment $[x_{2j-1}, x_{2j}]$, $\sqrt{p(x)} > 0$ for $x \in C_j$, $x_{2j} < x < x_{2j+1}$.*

Corollary. *If $p(x)$ is an even polynomial, then*

$$\lambda_{ni} - \lambda_{n,k-i+1} = O(n^{-\infty})$$

as $n \to \infty$.

Theorem 3. *If $p(x)$ is an even polynomial and has 4 zeros x_j, then as $n \to \infty$*

$$|\lambda_{n1} - \lambda_{n2}| = \frac{e^{-\lambda_{n1} c}}{a} \left(1 + O\left(\frac{1}{n}\right)\right),$$

$$a = \int_{x_1}^{x_2} \sqrt{|p(x)|}\, dx, \qquad c = \int_{x_2}^{x_3} \sqrt{p(x)}\, dx. \tag{4}$$

Formula (3), with accuracy up to $O(\lambda_n^{-2})$, was proved by Titchmarsh ([1]) for $p(z) = z^{2n} - 1$. Formula (4) was recently obtained in ([2]).

Let $p(z)$ be a polynomial, all zeros of $p(z)$ being simple. Denote

$$\xi(z, \lambda) = \lambda \int \sqrt{p(z)}\, dz.$$

Suppose that, for all $\arg\lambda$, the level lines $\operatorname{Re}\xi(z, \lambda) = \text{const}$ contain no more than two zeros of $p(z)$, and that for any fixed $\arg\lambda$ there is no more than one level line containing two zeros of $p(z)$. In ([3]) the notion was introduced of a complex K joining $+\infty$ and $-\infty$ (a complex of genus II). K contains exactly two zeros z_1, z_2 of the function $p(z)$.

Theorem 4. *Let $p(z)$ satisfy the conditions formulated above. Then there exists $\lambda_0 > 0$ such that for $|\lambda| > \lambda_0$ the discrete spectrum of problem (1) either is absent or is situated near some ray $\arg\lambda = \varphi_0$. The latter occurs only in the case when there exists a complex complex K joining $+\infty$ and $-\infty$, and as*

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$n \to \infty$ the asymptotics of λ_n is determined by formula (3), where the contour C encloses the zeros $p(z)$, $z_1, z_2 \in K$, and contains no other zeros of $p(z)$.

2. Let $p(z)$ be an entire function. A **Stokes line** is a maximal regular connected component of the line $\operatorname{Re}\xi(z,\lambda) = \text{const}$, one of whose ends is a zero of $p(z)$. Denote by $\Phi(\lambda)$ the collection of all Stokes lines for a given λ; by R_ε the z-plane from which nonintersecting neighborhoods of the zeros of $p(z)$ have been removed, whose images in the ξ-plane are disks of radius ε; and by $\alpha^+(z_0,\lambda)$ the curve joining the points z_0, ∞, along which the function $\operatorname{Re}\xi(z,\lambda)$ does not decrease and which is mapped by the function $\xi(z,\lambda)$ onto a broken line. $\alpha^-(z_0,\lambda)$ is the analogous curve along which $\operatorname{Re}\xi(z,\lambda)$ does not increase.

Let $p(z)$ satisfy the following conditions for all λ:

1) $\overline{\Phi}(\lambda) = \Phi(\lambda)$;

2) $\overline{\Phi}(\lambda) \setminus \overline{S(\lambda)} = \Phi(\lambda) \setminus S(\lambda)$ for any Stokes line $S(\lambda)$;

3)
$$\lim_{\substack{z\to\infty \\ z\in l(\lambda)}} \operatorname{Re}\xi(z,\lambda) = \infty$$
for any level line $l(\lambda) : \operatorname{Im}\xi(z,\lambda) = \text{const}$;

4)
$$\sup_{\lambda;\, z\in R_\varepsilon} \rho(z,\lambda) < \infty, \qquad \lim_{\substack{z\to\infty \\ z\in R_\varepsilon}} \rho(z,\lambda) = 0, \qquad \lim_{\substack{z\to\infty \\ z\in R_\varepsilon}} \frac{p'(z)}{[p(z)]^{3/2}} = 0;$$

5)
$$\sup_{\lambda;\, z\in R_\varepsilon} \rho_1(z,\lambda) < \infty, \qquad \lim_{\substack{z\to\infty \\ z\in R_\varepsilon}} \rho_1(z,\lambda) = 0;$$

Here
$$\rho(z,\lambda) = \inf_{\alpha^+(z,\lambda)} \int_{\alpha^+(z,\lambda)} |\delta(z)|\,|dz|,$$
$$\rho_1(z,\lambda) = \frac{|\delta(z)|}{|\sqrt{p'(z)}|} + \inf_{\alpha^+(z,\lambda)} \int_{\alpha^+(z,\lambda)} \left(\delta(t)\rho(t,\lambda) + \frac{d}{dt}\left(\frac{\delta(z)}{\sqrt{p(t)}}\right)\right) |dt|,$$
$$\delta(z) = \frac{[p'(z)]^2}{[p(z)]^{5/2}} + \frac{p''(z)}{[p(z)]^{3/2}}.$$

We note that examples of entire functions $p(z)$ for which conditions 1) or 2) are not fulfilled are unknown. Condition 3) is approximately equivalent to the following: $p(z)$ has no asymptotic value 0. Conditions 4), 5) require a certain regularity of the growth of $p(z)$ as $z \to \infty$.

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Theorem 5. *Let $p(z)$ be an entire function satisfying conditions 1)–5) as $z \to \infty$. If $p(z)$ satisfies the conditions of Theorem 1, then, as $n \to \infty$,*

$$\lambda_n \alpha = n\pi + \frac{\pi}{2} - \frac{i}{64\pi\alpha} \int_C \frac{[p'(z)]^2}{[p(z)]^{5/2}}\, dz + O(n^{-2}); \tag{5}$$

if, however, $p(z)$ satisfies the conditions of Theorem 2, then for the series λ_{nj} one has

instead of formula (5), with a replaced by a_j, where

$$a_j = \int_{x_{2j-1}}^{x_{2j}} \sqrt{|p(x)|}\, dx.$$

If $p(z)$ satisfies the conditions of Theorem 4 and conditions 1)–5) for all λ, then Theorem 4 holds, only for λ_n formula (3) holds with accuracy up to $O(\lambda_n^{-2})$.

Theorem 3 is also true for entire functions $p(z)$, but the conditions on $p(z)$ become very cumbersome.

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Moscow Institute of Physics and Technology

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Note: Figure translations are in progress. See original paper for figures.

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