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Abstract

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ON SYMMETRIC FLOWS OF A CONDUCTING FLUID ACROSS A MAGNETIC FIELD

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In the present work, plane and axially symmetric flows of a conducting compressible fluid are considered. A system of equations describing flows along equipotentials of the electric field is obtained. A class of solutions of these equations is considered, as well as solutions for flows in narrow channels.

For simplicity we shall assume that the fluid is inviscid and non-heat-conducting; then the stationary flow is described by the following equations of magnetohydrodynamics (^{1}):

$$\rho(\mathbf{v}\nabla)\mathbf{v} = -\nabla p + [\mathbf{jH}], \tag{1}$$

$$\operatorname{div}\rho\mathbf{v} = 0, \qquad \operatorname{div}\mathbf{H} = 0, \tag{2}$$

$$\operatorname{rot}\vec{\varepsilon} = 0, \qquad \rho T\mathbf{v}\nabla S = \nu_m j^2, \tag{3}$$

where

$$\vec{\varepsilon} \equiv \frac{c}{\sqrt{4\pi}}\mathbf{E} = \nu_m\mathbf{j} - [\mathbf{vH}] = \nabla\Phi, \qquad \mathbf{j} \equiv \operatorname{rot}\mathbf{H}, \qquad \nu_m = \frac{c^2}{4\pi\sigma},$$

ρ is the density, p the pressure, σ the conductivity, T the temperature, S the entropy, $\mathbf{v}$ the velocity, $\mathbf{B} \equiv \sqrt{4\pi}\mathbf{H}$ the magnetic field, and $\mathbf{E}$ the electric field. From (1)–(3) there follows the equation

$$\operatorname{div}\mathbf{q} = 0, \qquad \mathbf{q} = \rho\mathbf{v}\left(W + \frac{v^2}{2}\right) + [\vec{\varepsilon}\mathbf{H}], \tag{4}$$

which expresses conservation of the energy flux $\mathbf{q}$. Here W is the enthalpy, related to p and S by the thermodynamic relation

$$dW = \frac{dp}{\rho} + T\,dS. \tag{5}$$

Let us consider flow in an axially symmetric channel across an azimuthal magnetic field: $H = H_\varphi,\ v_\varphi = 0$. Introduce, for $\rho\mathbf{v}$ and $\mathbf{j}$, the stream functions ψ and $I = rH_\varphi$ according to the relations

$$\begin{pmatrix}\rho v_z\\ j_z\end{pmatrix} = \frac{1}{r}\frac{\partial}{\partial r}\begin{pmatrix}\psi\\ I\end{pmatrix}, \qquad \begin{pmatrix}\rho v_r\\ j_r\end{pmatrix} = -\frac{1}{r}\frac{\partial}{\partial z}\begin{pmatrix}\psi\\ I\end{pmatrix}. \tag{6}$$

We shall further require that the surfaces $\psi = \text{const.}$, on which the fluid streamlines lie, coincide with the equipotential surfaces $\Phi = \text{const.}$, i.e. $\Phi = \Phi(\psi)$. Then it follows from the first equation (3) that $(\nabla I \nabla \psi) = 0$, i.e. the fluid streamlines $\psi = \text{const}$ and the electric-current lines $I = \text{const}$ form orthogonal families of trajectories.

For the axially symmetric flows under consideration we obtain the following system of equations:

$$\frac{1}{\rho r}\frac{\partial}{\partial r}\frac{1}{\rho r}\frac{\partial \psi}{\partial r} + \frac{1}{\rho r}\frac{\partial}{\partial z}\frac{1}{\rho r}\frac{\partial \psi}{\partial z} + I\Phi''(\psi) + T\frac{\nabla S \nabla \psi}{\rho^2 r^2 v^2} - U'(\psi) = 0, \tag{7a}$$

$$W + \frac{v^2}{2} + I\Phi'(\psi) = U(\psi), \tag{7b}$$

$$\frac{I}{\rho r^2} - \nu_m \frac{\mathbf{v}\nabla I}{\rho r^2 v^2} = \Phi'(\psi), \tag{7c}$$

$$\rho T \mathbf{v} \nabla S = \frac{\nu_m}{r^2}(\nabla I)^2. \tag{7d}$$

The Bernoulli equation (7b) follows from equation (4), (7c) follows from the equality $\vec{\varepsilon} = -\nabla\Phi$, and (7a) follows from the Euler equation (1). In system (7) the unknowns are ψ, ρ, and I, while $\Phi(\psi)$ and $W(\psi)$ are arbitrary functions of ψ. If S is regarded as a function of ψ and I, then the term in (7a) proportional to T may be written in the form $T\partial S/\partial\psi$. Since equations (7) determine only the derivative $\partial S/\partial I$ along the streamlines $\psi = \text{const}$, S as a function of ψ may be prescribed arbitrarily and, in particular, one may set $\partial S/\partial \psi = 0$. For flows of an ideally conducting fluid ($\nu_m = 0$), system (7) reduces to that obtained in [2].

Similarly, plane flows independent of z, with $H = H_z,\ v_z = 0$, are described by the system of equations

$$\frac{1}{\rho}\frac{\partial}{\partial x}\frac{1}{\rho}\frac{\partial \psi}{\partial x} + \frac{1}{\rho}\frac{\partial}{\partial y}\frac{1}{\rho}\frac{\partial \psi}{\partial y} + H\Phi''(\psi) + T\frac{\nabla S \nabla \psi}{\rho^2 v^2} - U'(\psi) = 0, \tag{8a}$$

$$W + \frac{v^2}{2} + H\Phi'(\psi) = U(\psi), \tag{8b}$$

$$\frac{H}{\rho} - \nu_m \frac{\mathbf{v}\nabla H}{\rho v^2} = \Phi'(\psi), \tag{8c}$$

$$\rho T \mathbf{v}\nabla S = \nu_m (\nabla H)^2, \tag{8d}$$

where the stream function ψ is introduced by the relations $\rho v_x = \partial\psi/\partial y,\ \rho v_y = -\partial\psi/\partial x$, and the components $\mathbf{j}$ are determined in terms of H by the equalities $j_x = \partial H/\partial y,\ j_y = -\partial H/\partial x$. Equation (8d), taking (8b) and (8c) into account, can be transformed to the form

$$\rho \mathbf{v}\nabla \frac{v^2}{2} + \mathbf{v}\nabla\left(p + \frac{H^2}{2}\right) = 0. \tag{8e}$$

We shall seek a solution of (8), prescribing ρ as a known function of the velocity, $\rho = F'(v^2)$; then, according to (8e), we have

$$\frac{F}{2} + p + \frac{H^2}{2} = G(\psi). \tag{9}$$

Eliminating from (8b) and (9) the pressure $p = \varkappa\rho W$, where $\varkappa = (\gamma - 1)/\gamma$, we find $H = H(v^2)$:

$$H = \varkappa\Phi' F' \pm \sqrt{(\varkappa\Phi' F')^2 - F + 2G + \varkappa(v^2 - 2U)F'}. \tag{10}$$

From equation (8c) we obtain the quadrature determining the dependence of v on the arc length s along a streamline:

$$s = 2\nu_m \int \frac{H'(v^2)\,dv}{F(v^2) - F'(v^2)\Phi'(\psi)}. \tag{11}$$

To find the streamlines in the general case, it is necessary to solve equation (8a). In the simplest case $\Phi'' = 0,\ U' = 0,\ \partial S/\partial\psi = 0$, and $\rho = \text{const}$, the problem reduces to solving the Laplace equation $\Delta\psi = 0$ with the normal derivative $\partial\psi/\partial y$ prescribed by relation (11) at $y = 0$. If one restricts oneself to the approximation in which the parameters of the flow vary slowly along the x-axis, then, neglecting terms $\sim (\partial\psi/\partial x)^2$ and $\partial^2\psi/\partial x^2$, in the indicated case $\Phi'' = U' = \partial S/\partial\psi = 0$, from (8a) we obtain $\psi \simeq F'(v^2)vy$, where the velocity $v(x)$ may be approximately taken equal to $v(s)$, determined by equation (11).

If the distance f between the electrodes is small in comparison with the characteristic dimensions, then equations (7) reduce to the system

$$\rho r f v = \alpha = \text{const}, \tag{12a}$$

Fig. 1

Figure 1: Fig. 1

$$\frac{I}{\rho r^2} - \frac{\nu_m}{\rho r^2 v}\frac{dI}{ds} = \beta = \text{const}, \tag{12b}$$

$$W + \frac{v^2}{2} + \beta I = U = \text{const}, \tag{12 }$$

$$\rho v T\,\frac{dS}{ds} = \frac{\nu_m}{r^2}\left(\frac{dI}{ds}\right)^2 . \tag{12 }$$

Here ds is an element of the arc of the middle line $r = r(z)$ (see Fig. 1).

In the case under consideration of a narrow gap between ideally conducting electrodes $r_1(z)$ and $r_2(z)$, the requirement $\Phi = \Phi(\psi)$ is not a restriction, and equations (12) may be obtained directly from (1)–(4). Indeed, the first of equations (3), written as an integral around the contour $MNN'M'$, gives $\oint \vec{\varepsilon}\,dl = 0$, whence we obtain $\varepsilon_\perp f = \text{const}$, i.e., equation (12). The continuity equation (2) in integral form $\oint \rho v\,dS = 0$, where the integral is taken over the surface of the toroidal region bounded by the contour $MNN'M'$, leads to equation (12a). Similarly, equation (4) gives $q_\parallel r f = \text{const}$, since the energy flux q through the side walls is zero; hence, taking (12) into account, we obtain equation (12).

Fig. 1

Adding to system (12) the equation of state $p = p(\rho, T)$, we obtain 5 equations for 7 unknowns: ρ, p, T, v, r, f, I; consequently, two of these quantities may be prescribed arbitrarily. Let us replace equation (12) by the momentum equation

$$\rho v\,dv + \frac{1}{2r^2}\,dI^2 + dp = 0 \tag{13}$$

and prescribe the functions ρ and r in the form $\rho = F'(v^2)$, $r^{-2} = R'(I^2)$; then from (13) we obtain $F/2 + R/2 + p = G = \text{const}$. Eliminating from (12) the enthalpy $W = p/\varkappa\rho$ with the aid of the last relation, we obtain

$$\varkappa F'\left(\frac{v^2}{2} + \beta I - U\right) - \left(\frac{F}{2} + \frac{R}{2} - G\right) = 0. \tag{14}$$

Relation (14) determines $I = I(v^2)$. The expression for $v = v(s)$ is found by integrating (12):

$$s = 2\nu_m \int \frac{I'(v^2)\,dv}{I - \beta F'/R'}.$$

Differentiating (12) and eliminating dp and ds, we obtain an equation of the Hugoniot type

$$\left(v^2 - c_T^2 - \frac{\beta^* I}{1 - \nu_m I''/vI'}\right)\frac{dv}{v} = c_T^2 \frac{d(rf)}{rf} + \frac{\rho r^2 \beta\beta^*}{1 - \nu_m I''/vI'}\,\frac{d(f/r)}{f/r}, \tag{15}$$

where $c_T^2 = \rho(\partial W/\partial\rho)_S$ is the square of the speed of sound, $\beta^* = \beta + \gamma\nu_m I'/\rho r^2 v$. It follows from (15) that, in the acceleration of a cold plasma $p \ll H^2/2$, the determining quantity is f/r, while for the opposite sign of the inequality it is the quantity rf.

Plane flows in a narrow channel are described by the system of equations

$$\rho v f = \alpha = \text{const}, \tag{16a}$$

$$\frac{H}{\rho} - \frac{\nu_m}{\rho v}\frac{dH}{ds} = \beta = \text{const}, \tag*{(16)}$$

$$W + \frac{v^2}{2} + \beta H = U = \text{const}, \tag*{(16)}$$

$$\rho v T\,\frac{dS}{ds} = \nu_m \left(\frac{dH}{ds}\right)^2. \tag*{(16)}$$

These equations are analogous to (12), but the number of unknowns here is smaller by one. As before, set $\rho = F'(v^2)$; then the momentum equation $\rho v\,dv + d(p + H^2/2) = 0$ gives $F + 2p + H^2 = 2G$, and, eliminating p from this, with the aid of (16b), we find H as a function of v^2:

$$H = \chi\beta F' \pm \sqrt{(\chi\beta F')^2 - F + 2G + \chi(v^2 - 2U)F'}. \tag{17}$$

Let the middle plane of the channel be $y = 0$; then v as a function of x is determined by the integral

$$x = 2\nu_m \int \frac{H'(v^2)\,dv}{H - \beta F'(v^2)}.$$

For the one-dimensional problem, an analogous solution is given in the work (3).

In conclusion, we give the Hugoniot equation for plane flow, which is obtained from (16):

$$\left(v^2 - c_T^2 - \frac{\beta^* H}{1 - \nu_m H''/vH'}\right)\frac{dv}{v} = \left(c_T^2 + \frac{\rho\beta\beta^*}{1 - \nu_m H''/vH'}\right)\frac{df}{f}. \qquad (18)$$

Here $\beta^* = \beta + \gamma\nu_m H'/\rho v$. Equations (18) and (15) show that, for finite conductivity ($\nu_m \neq 0$), the velocity at which reversal of interactions occurs depends on the derivatives of the magnetic field along the velocity, H' and H''.

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[3] A. G. Kulikovskii and G. A. Lyubimov, *Magnetic Hydrodynamics*, Moscow, 1962.

Note: Figure translations are in progress. See original paper for figures.