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Abstract

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ON A GENERALIZATION OF THE RIEMANN-SCHWARZ SYMMETRY PRINCIPLE AND ITS APPLICATIONS

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The following Riemann–Schwarz symmetry principle for analytic functions is known (see, for example, ([1])). Let the function $F(z)$ be holomorphic in the domain D and continuous in $\overline{D}$. If the boundary of the domain D contains a segment of the real axis or an arc of the circle $|z| = 1$, and if

$$\operatorname{Re}[F(t)] = 0 \quad \text{for } t \in l, \tag{1}$$

(l denotes the segment or arc), then $F(z)$ can be analytically continued across l; namely, the function

$$\Phi(z) = \begin{cases} F(z), & z \in D, \\ -\overline{F(z_1)}, & z \in D^*, \end{cases}$$

is holomorphic in the domain $D + D^* + l$, where D^* is the reflection of the domain D with respect to l, and the point z_1 is the reflection of the point z with respect to l.

An analogous theorem holds for generalized analytic functions (see ([2])).

We generalize this theorem to the case in which condition (1) is replaced by the condition of the general Riemann–Hilbert problem.

Theorem. *Let $F(z)$ be holomorphic in the domain D and continuous in $\overline{D}$. If the boundary of the domain D contains a segment of a straight line or an arc of a circle, and if*

$$\operatorname{Re}[\lambda(t)F(t)] = 0 \quad \text{for } t \in l, \tag{2}$$

where l is a segment of a straight line or an arc of a circle, and $\lambda(t) \neq 0$ is a given function of Hölder class on l, then in D there exists a holomorphic function $\chi(z)$ such that $\chi(z)F(z)$ can be analytically continued across l; namely, the function

$$\Phi(z) = \begin{cases} \chi(z)F(z), & z \in D, \\ -\overline{\chi(z_1)}\,\overline{F(z_1)}, & z \in D^*, \end{cases} \tag{3}$$

is holomorphic in the domain $D + D^* + l$, where D^* is the reflection of the domain D with respect to l; z_1 is the reflection of z with respect to l; $\chi(z) \neq 0$ is a solution of the conjugation problem with boundary condition

$$\chi^+(t) = \frac{\lambda(t)}{\overline{\lambda(t)}}\,\chi^-(t), \quad t \in l,$$

with

$$\chi^-(t) = \overline{\chi^+(t)}.$$

As is known, $\chi(z)$ is constructed effectively (1), §83*.*

The proof of this theorem presents no difficulty, and we omit it.

We shall call the function $\chi(z)$ the **factor of analytic continuation** of the function $F(z)$.

If $\lambda(t)$ is a real function, then, obviously, one may take $\chi(z) \equiv 1$, and from (1) we obtain the usual symmetry principle.

If condition (2) is satisfied for a generalized analytic function, then, correspondingly, one can define an analytic multiplier for the continuation of a generalized analytic function in the sense of I. N. Vekua (see (2)).

In the work (3), A. V. Bitsadze reduced the solution of the Tricomi problem for an equation of M. A. Lavrent' ev type to the Riemann–Hilbert problem for the half-plane with boundary conditions

$$\mathrm{Re}[F(t)] = \varphi(t) \quad \text{for } \left|t - \frac{1}{2}\right| = \frac{1}{2},\ y > 0,\ t = x + iy,$$

$$\mathrm{Re}[(1 - i)F(x)] = \psi(x) \quad \text{for } 0 \leq x \leq 1,$$

and, applying the symmetry principle, constructed an effective solution.

Applying our generalized symmetry principle and following an analogous path, we can effectively solve the general Riemann–Hilbert problem for many domains of a particular kind, for example, for the half-disk, for a quarter of a disk, for a quadrant, and in general for a sector of a disk with aperture $\pi/2^n$ for an angle $\pi/2^n$. All these problems are solved by a single method.

Let us consider, for example, the Riemann–Hilbert problem in the case of a half-disk: find a function $F(z)$, holomorphic in the half-disk D ($|z| < 1,\ y >$

$0,\ z = x + iy)$, continuous in $\overline{D}$ except, perhaps, at the points $x = 1,\ x = -1$, satisfying the boundary conditions:

$$\begin{aligned} \mathrm{Re}[\lambda(t)F(t)] &= \varphi(t), \quad \text{for } t \in \sigma, \\ \mathrm{Re}[\mu(x)F(x)] &= \psi(x), \quad \text{for } -1 \le x \le 1, \end{aligned} \tag{4}$$

where σ is the semicircle $|t| = 1,\ y > 0,\ t = x + iy$, and $\lambda(t) \ne 0,\ \mu(x) \ne 0$ are given functions of Hölder class on the corresponding lines. In a neighborhood of each of the points $x = -1,\ x = 1$, the required function is subject to the condition

$$F(z) \le \frac{\text{const}}{|z \pm 1|^{\alpha}}, \qquad \alpha < 1.$$

To solve this problem, represent $F(z)$ in the form of the sum of two functions $F_1(z), F_2(z)$, holomorphic in the domain D, which are respectively solutions of the Riemann-Hilbert problems with boundary conditions

$$\mathrm{Re}[\lambda(t)F_1(t)] = \varphi(t) \quad \text{for } t \in \sigma, \tag{5}$$

$$\mathrm{Re}[\mu(x)F_1(x)] = 0 \quad \text{for } -1 \le x \le 1, \tag{6}$$

$$\mathrm{Re}[\lambda(t)F_2(t)] = 0 \quad \text{for } t \in \sigma, \tag{7}$$

$$\mathrm{Re}[\mu(x)F_2(x)] = \psi(x) \quad \text{for } -1 \le x \le 1. \tag{8}$$

It is obvious that $F(z)$ satisfies condition (4).

Using (6), one can define, according to formula (3), a function $\Phi(z)$, holomorphic inside the disk $|z| < 1$. From condition (5) we conclude that $\Phi(z)$ is a solution of the Riemann-Hilbert problem for the disk, with boundary conditions

$$\mathrm{Re}\left[\frac{\lambda(t)}{\chi^{+}(t)}\Phi(t)\right] = \varphi(t), \qquad t \in \sigma,$$

$$\mathrm{Re}\left[\frac{\overline{\lambda(\bar{t})}}{\chi^{-}(t)}\Phi(t)\right] = -\varphi(\bar{t}), \qquad t \in \bar{\sigma},$$

where $\bar{\sigma}$ is the reflection of the semicircle σ with respect to the real axis. A solution of the given class of this last problem, if it exists, is constructed, as is known (1), effectively. The desired function $F_1(z)$ in the given half-disk, by virtue of (3), will be $\Phi(z)/\chi(z)$.

The construction of $F_2(z)$ is reduced in an analogous way to the solution of the Riemann–Hilbert problem for the half-plane.

For a quadrant, the solution of the Riemann–Hilbert problem can also be represented as the sum of two holomorphic functions, one of which satisfies a homogeneous condition for $x > 0,\ y = 0$ and an inhomogeneous condition for $y < 0,\ x = 0$, while the other satisfies analogous conditions. Applying the generalized symmetry principle, one can reduce the problem to two Riemann–Hilbert problems for the half-plane. Thus, the problem under consideration is solved effectively.

In general, the Riemann–Hilbert problem for the domains mentioned above is solved effectively in an analogous way, i.e., by reduction to the problem for the disk or the half-plane.

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References

[1] N. I. Muskhelishvili, *Singular Integral Equations*, Moscow—Leningrad, 2nd ed., 1962. [2] I. N. Vekua, *Generalized Analytic Functions*, Moscow, 1959. [3] A. V. Bitsadze, *Equations of Mixed Type*, Publishing House of the Academy of Sciences of the USSR, 1959.

Note: Figure translations are in progress. See original paper for figures.