THE METHOD OF VARIATIONS IN THE THEORY OF QUASICONFORMAL MAPPINGS OF CLOSED RIEMANN SURFACES

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Abstract

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THE METHOD OF VARIATIONS IN THE THEORY OF QUASICONFORMAL MAPPINGS OF CLOSED RIEMANN SURFACES

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Various variational formulas connected with quasiconformal mappings of Riemann surfaces have been applied by many authors ([3–7]). In the present note, by extending the method of ([2]), variational formulas are derived for closed Riemann surfaces. These formulas are then applied to the solution of extremal problems for quasiconformal mappings and to the local study of the Teichmüller space T_g ([3–6]).

1. Let S be a closed Riemann surface of genus $g > 1$, conformally equivalent to $\Pi = U/\Gamma$, where U is the disk $|z| < 1$, and Γ is a Fuchsian group of fractional-linear transformations of U onto itself, isomorphic to the fundamental group of the surface S. The case $g = 1$ also fits into the general scheme, but is not considered here. The following considerations can also be carried out for a compact surface with boundary, only in this case the group Γ, together with fractional-linear transformations, contains transformations conjugate to fractional-linear ones.

By a variation of the surface S we shall mean a function $w = f(z)$ mapping Π onto some polygon U/Γ^* (where Γ^* is a Fuchsian group), normalized and having the property that the mapping effected by it is conformal outside some disk $K : |z - \zeta| < r,\ K \subset \Pi$, while inside it has constant characteristic h. Thus the variation induces a quasiconformal mapping of the disk $|z| \le 1$ onto the disk $|w| \le 1$, with $f(Az) = A^* f(z),\ A \in \Gamma,\ A^* \in \Gamma^*$. Fixing, for example, $f(0) = 0,\ f(1) = 1$, and putting

$$W(z, \zeta, \Gamma) = \frac{1}{\pi} \sum_{A \in \Gamma} \frac{z(1-z)}{A\zeta(1 - A\zeta)(z - A\zeta)} \left(\frac{dA\zeta}{d\zeta} \right)^2,$$

$$W_1(z, \bar{\zeta}, \Gamma) = \frac{1}{\pi} \sum_{A \in \Gamma} \frac{z(1-z)}{A\bar{\zeta}(1 - A\bar{\zeta})(1 - zA\bar{\zeta})} \left(\frac{dA\bar{\zeta}}{d\zeta} \right)^2,$$

we find that the local variation is expressed by the formula:

$$w = z + d\sigma_\zeta \left[hW(z,\zeta,\Gamma) + \bar{h}W_1(z,\bar{\zeta},\Gamma)\right] + O[(h\,d\sigma_\zeta)^2], \tag{1}$$

where $d\sigma_\zeta = \pi r^2$, $|A^{-1}z - \zeta| > r$, while inside the disk $|A^{-1}z - \zeta| < r$, $A \in \Gamma$, the characteristic $h(z) = h\exp[2i\arg A'(\zeta)] + O(r)$. To the variation $w = f(z)$ there corresponds a change of the moduli of the surface S ([4,7]):

$$\delta\tau_{ij} = \tau^*_{ij} - \tau_{ij} = -2ihd\sigma_\zeta\,\theta_i(\zeta)\theta_j(\zeta) + o(hd\sigma_\zeta), \tag{2}$$

where τ_{ij} and τ^*_{ij} are, respectively, the moduli of the surfaces S and $S^* = U/\Gamma^*$, and $\theta_1(z)\,dz, \ldots, \theta_g(z)\,dz$ is a normalized basis of Abelian differentials of the first kind on S. The classical theorem of M. Noether asserts that, if the surface S is nonhyperelliptic, then the number of linearly independent products $\theta_i\theta_j$ is $3g-3$ (over the field of complex numbers), and in the case of a hyperelliptic surface is equal to $2g-1$.

For mappings close to conformal ones, the corresponding integral variational formulas are valid, which are written as integrals of the right-hand sides of (1) and (2) over the fundamental polygon Π. In this case the estimate of the remainder term depends on the smoothness of $h(z)$.

2. For mappings with arbitrary measurable characteristics

$$h(z), \quad |h(z)| \le h_0 < 1, \quad h(z) = -\frac{p(z)-1}{p(z)+1}e^{2i\theta(z)}$$

(where $p(z)$ and $\theta(z)$ are characteristics in the sense of M. A. Lavrent' ev [1]), the analogue of the method of variations is the well-known method of parametric representation of quasiconformal mappings.

Let the function $w = f(z)$ map quasiconformally the surface $S = U/\Gamma$ onto $S^* = U/\Gamma^*$ (i.e. the disk U onto itself so that $f(Az) = A^*f(z)$, $A \in \Gamma$, $A^* \in \Gamma^*$) with characteristic $h(z) = f_{\bar{z}}/f_z$, where $f(0) = 0$, $f(1) = 1$, and let the mapping of S onto S^* be homotopic to the identity. Denote by S_t the surfaces obtained from S under the mappings $w = f(z,t)$ with characteristics $h(z,t) = th(z)$, $0 \le t \le 1$, and with the same normalization, and let Γ_t be the Fuchsian group corresponding to S_t, $\Pi_t = U/\Gamma_t$. Then the following holds.

Theorem 1. The function $w = f(z,t)$ satisfies the equation

$$\frac{\partial w}{\partial t} = \iint_{\Pi_t} \left[\varphi(\zeta,t)W(w,\zeta,\Gamma_t) + \overline{\varphi(\zeta,t)}\,W_1(w,\bar{\zeta},\Gamma_t)\right] d\sigma(\zeta) \tag{3}$$

and the initial condition $f(z,0) = z$; here

$$\varphi(\zeta,t) = h(f^{-1}(\zeta,t))\exp\left[-2i\arg\left(\frac{\partial f^{-1}(\zeta,t)}{\partial t}\right)\right] \Big/ \left(1 - |h(f^{-1}(\zeta,t))|^2 t^2\right).$$

For a given function $\varphi(\zeta,t)$ and initial condition $z \in \Pi$, the solution of equation (3) is unique and therefore coincides with $w = f(z,t)$.

3. Denote by $V_q(\Gamma)$ the class of quasiconformal mappings $w = f(z)$ of the disk $|z| \leq 1$ onto the disk $|w| \leq 1$ with characteristic $p(z) \leq q$ such that $f(Az) = A^* f(z)$, $A \in \Gamma$, $A^* \in \Gamma^*$, where Γ^* are Fuchsian groups of the first kind, with normalization $f(0) = 0$, $f(1) = 1$ (i.e. $V_q(\Gamma)$ is the class of quasiconformal mappings of the closed Riemann surface $S = U/\Gamma$ onto the surfaces $S^* = U/\Gamma^*$).

We consider the following two general problems.

Problem 1. In the class $V_q(\Gamma)$, find the maximum of the real continuously differentiable function

$$F(z_1, \dots, z_n, w_1, \dots, w_n), \quad w_k = w(z_k) = u_k + iv_k.$$

The solution of this problem is given by the following theorem.

Theorem 2. There exists a Fuchsian group Γ_1 of conformal self-mappings of the disk $|w| \leq 1$ such that the quasiconformal mapping

$$w = f(z) \in V_q(\Gamma)$$

with characteristic of the inverse mapping $h(w)$, satisfying the relations

$$|h(w)| = h_0 = (q-1)/(q+1),$$

$$\arg h(w) = -\arg \sum_{k=1}^{n} \left[F_{w_k} W(w_k, w, \Gamma_1) + F_{\bar{w}_k} W_1(\bar{w}_k, \bar{w}, \Gamma_1) \right],$$

is an extremal mapping for which the maximum of the real function

$$F(z_1, \dots, z_n, w_1, \dots, w_n), \quad w_k = w(z_k),$$

is attained in the class $V_q(\Gamma)$.

The proof is carried out according to the scheme applied in [2] for the case of the disk.

Problem 2. Find the set of values $D(\Phi)$ of the continuous functional

$$\Phi = \Phi(z_1, \dots, z_n, w_1, \dots, w_n), \quad w_k = w(z_k) = u_k + iv_k,$$

defined on the class $V_q(\Gamma)$.

From the compactness of the class $V_q(\Gamma)$ and Theorem 1 it follows that $D(\Phi)$ is a bounded closed connected set. Following the terminology adopted

in the theory of univalent analytic functions ([8]), we shall call functions $w = f(z) \in V_q(\Gamma)$ boundary functions if they take boundary points into the set $D(\Phi)$. Knowing the boundary $D(\Phi)$, one can also find the set $D(\Phi)$ itself. We shall assume that the function Φ is continuously differentiable with respect to u_k, v_k.

Theorem 3. The mappings $z = f^{-1}(w)$, inverse to the boundary mappings, have characteristics $h(w)$ determined by the formulas:

$$|h(w)| = h_0 = (q-1)/(q+1),$$

$$\arg h(w) = -\arg \sum_{k=1}^{n} \left[e^{-i\alpha} \Phi_{w_k} W(w_k, w, \Gamma_\alpha) + e^{i\alpha} \overline{\Phi}_{w_k} W_1(\overline{w_k}, w, \Gamma_\alpha) \right] + \pi.$$

Here Γ_α are certain Fuchsian groups of fractional-linear transformations of the disk $|w| < 1$ onto itself, and α is an arbitrary number in $[0, 2\pi]$.

The assertion of the theorem is obtained by reducing Problem 2 to the determination of functions $w = f(z) \in V_q(\Gamma)$ for which

$$\min_{\Phi_0 \in D(\Phi)} |\Phi_0 - a|$$

is attained, $a \notin D(\Phi)$.

Theorems 2 and 3 give a solution of the extremal problems posed in terms of the characteristic $h(w)$ and the unknown Fuchsian groups of the first kind. If the quantity $\varepsilon = q - 1$ is small, then one can use integral variational formulas for the approximate determination of the required mapping functions and obtain a more effective solution of the problem.

4. Instead of the disk U, let us consider the upper half-plane $H : \operatorname{Im} z > 0$. If the surface S is nonhyperelliptic, then formula (2) makes it possible to find $3g - 3$ linearly independent (over the field of complex numbers) Beltrami differentials $\mu_l(z)\, d\bar{z}/dz$, which form a basis in the space $B(\Gamma)/N(\Gamma)$ ([5],[6]) and have the property that the corresponding variations change only one of the moduli of the surface S (up to quantities $o(\varepsilon)$, $\varepsilon \to 0$). For this the surface S must be marked simultaneously at $3g - 3$ points $\zeta_l \in H/\Gamma$, which are chosen so that

$$\det \|\theta_i(\zeta_l)\theta_j(\zeta_l)\| \neq 0,$$

where the indices (i, j) correspond to linearly independent products $\theta_i\theta_j$. For hyperelliptic surfaces such a basis is determined with the aid of certain relations from ([4]). Then, using the results of ([5]), one can find an explicit expression (in terms of the group Γ) for the metric of Teichmüller space T_g at each point $S \in T_g$, as well as explicit expressions for the components of the curvature tensor, the Ricci tensor, and the scalar curvature of the space T_g.

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Note: Figure translations are in progress. See original paper for figures.