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Abstract

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EFFECTIVE ERROR ESTIMATES FOR SOLUTIONS BY THE GRID METHOD OF THE DIRICHLET PROBLEM FOR THE LAPLACE EQUATION ON POLYGONS

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Known error estimates for the solution by the grid method of the Dirichlet problem for the Laplace equation, converging to the exact solution with rate h^k, where h is the mesh step, $1 \leq k \leq 6$, have been obtained, in particular, under the condition of boundedness of the k-th derivatives of the solution on the closed domain ($^{1-3}$). If the domain is a polygon M with an angle β, $\pi/k < \beta < \pi/(k-1)$, then, for arbitrarily prescribed boundary values that are as smooth as desired on the sides and continuous at the corner points, the k-th derivatives of the solution, as a rule, are unbounded on $\overline{M}$ (^{4}). This introduces a specificity into the construction of difference equations on a polygon and into the obtaining of error estimates. In the present note, estimates are first given for the derivatives of a harmonic function on a polygon and for the coefficients of the asymptotic representation of a harmonic function at the corners, expressed explicitly in terms of the maxima of the moduli of the derivatives of the function on the sides of the polygon. The estimates are obtained, relying on the results of (^{5}), by methods of the theory of functions of a complex variable. Next, two types of error estimates are proposed for an approximate solution of the Dirichlet problem for the Laplace equation, obtained on a grid in the form of a solution of a system of difference equations composed of ordinary difference operators and special operators that take into account the asymptotics of the harmonic function at the corners. The first estimates are analytic, of Gershgorin-estimate type, but expressed in terms of known quantities. The other estimates, more accurate, are obtained in the form of a solution of an auxiliary system of difference equations. The method considered for solving the Dirichlet problem on a polygon, with sufficiently smooth boundary values, makes it possible to obtain an approximate solution on the grid converging to the exact solution with rate $h^{m-2-\varepsilon}$, where $4 \leq m \leq 8$.

1. Let a polygon M be given with boundary Γ, having N sides. Let Γ_j, $j = 1, 2, \ldots,$—be the sides, including the corner points, numbered counterclockwise; $\alpha_j\pi$ are the angles between Γ_j and Γ_{j+1}, $0 < \alpha_j \leq 2$; $P_j = \Gamma_j \cap \Gamma_{j+1}$.

We shall say that $f \in C_m(E)$ if f is m times continuously differentiable on E. Consider the boundary-value problem

$$\Delta u(x, y) = 0 \text{ on } M, \qquad u = \varphi_j(s) \text{ on } \Gamma_j, \qquad j = 1, 2, \dots, N, \tag{1}$$

where $\varphi_j \in C_\nu(\Gamma_j)$, $\nu \geq 0$. Denote $\Phi_j^k = \sup_{\Gamma_j} |\varphi_j^{(k)}|$; $\Phi = \max_j \Phi_j^0$; $\widetilde{\Phi}_j^k = \max\{\Phi_j^k, \Phi_{j+1}^k\}$;

$$U^k(E) = \max_{\alpha+\beta=k} \sup_{0 \leq \theta < \pi/2} \sup_E \left|\partial^{\alpha+\beta} u / \partial x_\theta^\alpha \, \partial y_\theta^\beta\right|^*,$$

where x_θ, y_θ are the variables in a coordinate system rotated through an angle θ; R_j is a rectangle with one side $\gamma_j \subset \Gamma_j$ of length b, and the second side of length a_j, with the distances between the endpoints of γ_j and Γ_j not less than a_j, and half of every circle of radius a_j with center on R_j, lying above the diameter parallel to Γ_j, in the inward direction— of the normal to Γ_j, belongs to $\overline{M}$; $S_j(r)$ is the sector of a circle of radius r with center P_j, formed by the sides Γ_j and Γ_{j+1}; $T_j(r)$ is the arc of the sector $S_j(r)$; M_δ is the set of points M whose distance from Γ is greater than δ.

Lemma 1. The inequality (2) holds

$$U^\mu(\overline{M}_\delta) \leq \mu! 4\Phi / \pi\delta^\mu, \qquad \mu \geq 0. \tag{2}$$

Lemma 2. If $\varphi_j \in C_{\mu+1}(\Gamma_j)$, then

$$U^\mu(\overline{R}_j) \leq \frac{2 + 2^{\mu+3/2}}{\pi(\mu+1)} a_j \Phi_j^{\mu+1} + 4\,\frac{\mu!}{\pi}\left(\frac{\Phi}{a_j^\mu} + \sum_{k=0}^{\mu} \frac{2^k \Phi_j^k}{k!\, a_j^{\mu-k}}\right) + \sqrt{2}\,\Phi_j^\mu. \tag{3}$$

Let P_j be the origin, and let the x-axis be directed along Γ_{j+1}; $z = x + iy, \rho = |z|; \theta = \arg z$; $1 < \sigma \leq \nu$; $\varphi_t \in C_\nu(\Gamma_t)$; $t = j, j+1$;

$$f_{j\sigma} = \begin{cases} c_{j0}\theta + \displaystyle\sum_{n=0}^{\sigma-1} d_{jn} \operatorname{Re} z^n + \sum_{n=1}^{\sigma-1} e_{jn} \operatorname{Im} z^n, & \alpha_j \text{ irrational}, \\ \displaystyle\sum_{k=0}^{\sigma_*}{}' c_{jk} \operatorname{Im}\{z^{kq_j} \ln z\} + \sum_{n=0}^{\sigma-1} d_{jn} \operatorname{Re} z^n + \sum_{n=1}^{\sigma-1} e_{jn} \operatorname{Im} z^n, & \alpha_j = p_j/q_j, \end{cases}$$

where $\sigma_* = [(\sigma-1)/q_j]$; p_j/q_j is an irreducible fraction; $\sum'$ is a sum not extended to $n \equiv 0 \pmod{q_j}$; c_{jn}, d_{jn}, e_{jn} are numbers determined by the conditions: $\varphi_t - f_{j\sigma}|_{\Gamma_t} = o(\rho^{\sigma-1})$; $t = j, j+1$.

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* Below, analogous upper bounds for derivatives of functions denoted by the corresponding lowercase letters are denoted by other capital Latin letters.

Denote:

$$\omega_j(k) = \max_{\sum_p k_p = k} \prod_p \left(\prod_{q=0}^{k_p-1} |\alpha_j - q| \right); \qquad \Lambda_j^{\chi} = 4\,\frac{\Phi + F^0_{j\nu_j}(T_j(r_j))}{\pi + r_j^{\chi/\alpha_j}(1-\lambda_j^{1/\alpha_j})\chi} +$$

$$+\widetilde{\Phi}_j^{\nu_j} r_j^{\nu_j - \chi/\alpha_j} \left(\frac{4}{\pi(1-\lambda_j^{1/\alpha_j})\chi} \sum_{k=0}^{\chi} \frac{\omega_j(k)}{(\nu_j - k)!} + \frac{2 + 2^{\chi+3/2}}{(\nu_j - \chi - 1)!\pi} \omega_j(\chi+1) + \frac{\sqrt{2}\,\omega_j(\chi)}{(\nu_j - \chi)!} \right),$$

$$\Psi_j^{\mu}(\chi, \rho) = 4\rho^{\chi/\alpha_j - \mu} \mu!\, \frac{(1+\tau_j)^{\chi/\alpha_j}}{\pi \tau_j^{\mu}} \Lambda_j^{\chi} +$$

$$+\rho^{\nu_j - \mu} \widetilde{\Phi}_j^{\nu_j} \left(\frac{(2 + 2^{\mu+3/2}\tau_j(1+\tau_j)^{\nu_j - \mu - 1})}{(\nu_j - \mu - 1)!(\mu+1)\pi} + \frac{\sqrt{2}}{(\nu_j - \mu)!} + \frac{\mu!}{\pi} \sum_{k=0}^{\mu} \frac{2^{k+2}}{(\nu_j - k)!\, k!\, \tau_j^{\mu-k}} \right).$$

Lemma 3. Let $r_j > 0$; $S_j(r_j) \subset \overline{M}$; $\sigma(k) = [(k+1)/\alpha_j]$; $\chi_j \geq 1$; $m \geq 0$; $\nu_j = \max\{\sigma(\chi_j) + 1, \chi_j + 1, m + 1\}$; $\varphi_t \in C_{\nu_j}(\Gamma_t)$; $t = j, j+1$; $0 < \lambda_j < 1$; $\tau_j = \min\{\sin(a_j/2),\ 1/\lambda_j - 1\}$ for $\alpha_j < 1/2$; $\tau_j = \min\{\sqrt{2}/2,\ 1/\lambda_j - 1\}$ for $\alpha_j \geq 1/2$; $r_j^* = r_j\lambda_j/(1+\tau_j)$; then on $S_j(r_j^*)$

$$u = f_{j\nu_j} + g_j = f_{j\nu_j} + v_j + w_j; \tag{4}$$

$$v_j = \sum_{k=1}^{\chi_j - 1} \beta_{jk} \rho^{k/\alpha_j} \sin\frac{k\theta}{\alpha_j}, \quad (\chi_j > 1); \qquad v_j = 0 \quad (\chi_j = 1); \tag{5}$$

$$|\beta_{jk}| \leq \frac{(2^k + 2k(\alpha_j\sigma(k) - k))\, r_j^{\sigma(k) - k/\alpha_j}}{(\sigma(k))!\, k(\alpha_j\sigma(k) - k)\pi} \widetilde{\Phi}_j^{\sigma(k)} + 4\,\frac{\Phi + F^0_{j\sigma(k)}(T_j(r_j))}{\pi r_j^{k/\alpha_j}}; \tag{6}$$

$$G_j^{\mu}(T_j(\rho)) \leq \Psi_j^{\mu}(1, \rho), \qquad W_j^{\mu}(T_j(\rho)) \leq \Psi_j^{\mu}(\chi_j, \rho), \qquad 0 \leq \mu \leq m. \tag{7}$$

2. Let $S_j(r_j) \subset \overline{M}$; $0 < \lambda_j < 1$; $\varkappa_j \geqslant 1$; $m = 4, 8$; $\bar{\nu}_j = \max\{\nu_{j-1}, \nu_j\}$ $(\nu_0 = \nu_N)$; $\varphi_i \in C_{\nu_j}(\Gamma_j)$, $j = 1, 2, \dots, N$; $\delta > 0$; $M \subset \bigcup_{j=1}^{N} R_j \cup S_j(r_j^*/2) \cup M_\delta$.

Construct on $\overline{M}$ a square grid with mesh size h $(^3)$, for which one can choose numbers $r_{j\tau}$, $\tau = 0, 1, 2$, such that $2\varkappa_j h \leqslant r_{j0} < r_{j1} - h < r_j^* - 2h$; $r_j^*/2 + \chi_l^h \leqslant r_{j2} \leqslant r_j^* - \chi_l^h$; $r_{j0} \leqslant r_{j2}$, $j = 1, 2, \dots, N$; $\chi_l^h = h\max\{\sqrt{2}, t_l\}$, where t_l is a quantity independent of h, defined below, $S_p(r_{p2}) \cap S_q(r_{q2}) = \emptyset$ for $p \neq q$, and additional requirements following from what follows are satisfied. Denote by Π_j^h the set of grid nodes on $S_j(r_{j0})$; by Ω_j^h, the set of nodes on $S_j(r_{j1}+h) \setminus S_j(r_{j1}-h)$; by $M_{m_j}^h$, the set of nodes on $S_j(r_{j2}) \setminus S_j(r_{j0})$ which, together with the minimal square $\overline{E}_m$ containing the m nearest nodes, belong to $\overline{M}$; by $\Gamma_{m_j}^h$, the set of the remaining nodes on $S_j(r_{j2}) \setminus S_j(r_{j0})$; by M_m^h, the set of nodes on $M \setminus \bigcup_{j=1}^{N} S_j(r_{j2})$ which, together with $\overline{E}_m$, belong to $\overline{M}$; and by Γ_m^h, the set of the remaining nodes on M.

Consider the system of difference equations

$$u_h = A_m u_h \quad \text{on } M_m^h; \qquad u_h = B_l u_h \quad \text{on } \Gamma_m^h;$$

$$u_h = f_{j\nu_j} + A_m(u_h - f_{j\nu_j}) \quad \text{on } M_{m_j}^h; \qquad u_h = f_{j\nu_j} + B_l(u_h - f_{j\nu_j}) \quad \text{on } \Gamma_{m_j}^h; \tag{8}$$

$$u_h = f_{j\nu_j} + D_{\varkappa_j}(u_h - f_{j\nu_j}) \quad \text{on } \Pi_j^h, \qquad j = 1, 2, \dots, N,$$

where $m = 4, 8;\ l \leqslant m;\ A_m$ is the known averaging operator over m points (^{3}), with

$$|u - A_4 u| \leqslant h^4 U^4(\overline{E}_4)/24; \qquad |u - A_8 u| \leqslant h^8 U^8(\overline{E}_8)/7!2;$$

B_l is an interpolation operator (for $l \leqslant 5$, see (^{6})) such that

$$|u - B_{lu}| \leqslant h^l C_l U^l(\overline{E}_l'),$$

where $\overline{E}_l'$ is the minimal convex closed set containing the interpolation points;

$$C_l = \max_{\Gamma_h} \sum |d_l|\, t^l/l!h^l; \qquad \Gamma_h = \bigcup_{j=1}^{N} \Gamma_{m_j}^h \cup \Gamma_m^h;$$

t is the distance to an interpolation node; $t \leqslant ht_l$; t_l is a constant independent of the choice of grid; d_l is an interpolation coefficient, with $\sum' |d_l| \leqslant 1 - \varepsilon_l;\ \varepsilon_l > 0$; $\sum$ is the sum over all interpolation points, and $\sum'$ is the sum over interpolation points belonging to M; $D_{\varkappa_j}$ is an operator satisfying the condition

$$D_{\varkappa_j} v_j(P) \equiv \sum_{\Omega_j^h} a_{PQ} v_j(Q) = v_j(P),$$

where v_j is a function of the form (5) with arbitrary coefficients β_{jk}, $P \in \Pi_j^h$, $Q \in \Omega_j^h$, and r_{j0} is such that

$$\max_{P \in \Pi_j^h} \sum_{\Omega_j^h} |a_{PQ}| \leqslant 1 - \delta_j, \qquad \delta_j > 0.$$

For sufficiently small h, the existence of the operators $D_{\varkappa_j}$ is obvious. Indeed, the coefficients β_{jk} of the function v_j are determined as the solution of a system of equations with determinant nonzero and with free terms that are the values of v_j at the points dividing $T_j(r_{j1})$ into $\varkappa_j$ equal parts. The determinant remains bounded below in modulus by a positive quantity in a finite neighborhood of the indicated points, in which, for small h, grid nodes will be found. In practice, the β_{jk} are expressed through the values of v_j at nodes nearest to the uniformly spaced points on an arc of radius of order $h\varkappa_j/\alpha_j$. Then, using (5), one computes a_{PQ} and determines a value r_{j0} of order r_{j1} and a set Π_j^h on which

$$\sum_{\Omega_j^h} |a_{PQ}| \leqslant 1 - \delta_j.$$

Denote

$$\xi_m = \max\left\{U^m(\overline{M}_\delta),\ \max_j U^m(\overline{R}_j),\ \max_l G_l^m(S_j(r_j^*) \setminus S_j(r_{j0} - \sqrt{2}\,h))\right\},$$

$$\eta_l = \max\{U^l(\overline{M}_\delta),\ \max_j U^l(\overline{R}_j),\ \max_j G_j^l(S_j(r_j^*) \setminus S_j(r_{j0} - ht_l))\},$$

$$\vartheta = \max_j((r_{j1} + h)^{\varkappa_j/\alpha_j}(1 - \delta_j) + r_{j0}^{\varkappa_j/\alpha_j})\Lambda_j^{\varkappa_j}/\delta_j.$$

Lemma 4. The inequality holds

$$|u_h - u| \leqslant \vartheta + h^l C_l \eta_l/\varepsilon_l + h^{m-2} C'_m \rho_0^2 \xi_m/\varepsilon^*, \tag{9}$$

where $\varepsilon^* = \min\{\varepsilon_l, \min_j \delta_j\}$; ρ_0 is the radius of a circle containing M; $C'_4 = 1/4!$; $C'_8 = 30/9!$.

Using (2), (3), (7), we estimate the right-hand side of inequality (9) in terms of known quantities. Moreover, if $l = m - 2$, $0 < ah^{\theta_j} < r_{j0}$, $r_{j1} < bh^{\theta_j}$, $j = 1, 2, \dots, N$, where $\theta_j = \alpha_j(m-2)/(\varkappa_j - 1 + \alpha_j m)$, then, according to (2), (3), (7), and (9), $u_h - u = O(h^{\mu_0})$, where $\mu_0 = \min_j \varkappa_j(m-2)/(\varkappa_j - 1 + \alpha_j m)$.

Using the methods of § 6 (^{2}) and (7), one can show that for

$$\theta_j = \alpha_j(m-2)/(\varkappa_j - 1 + \alpha_j(m-2))$$
$$\mu_0 = \min\{m-2, \min_j \varkappa_j(m-2)/(\varkappa_j - 1 + \alpha_j(m-2))\}.$$

3. A more accurate error estimate than (9) is a majorant obtained as the solution of a system of equations differing from (8) in that the boundary values are taken equal to zero, the signs are changed for the negative coefficients of the operators D_{x_j} and B_l, and into each equation there is introduced a free term which is the smallest possible estimate of the residual u for the corresponding equation (8). The compilation of equations (8), as well as the choice of r_{j1} for which, on a fixed grid, the error estimate is minimal, can be automated on a computer. The selection of r_{j1} can be carried out independently, in view of the weak mutual influence of the errors at the angles (see, for example, (^{8})).

4. At angles $\alpha_j = 1/q_j$, which possess special properties (see (^{9,4})), v_j is a polynomial estimated by inequality (6). At these angles, when $\varkappa_j/\alpha_j \geqslant m^*$ ($m^* = m$ for $\alpha_j > 1$; $m^* = m - 2$ for $\alpha_j \leqslant 1$), one may put $r_{j0} = 0$ and construct difference equations by means of the operators A_m and B_l. In (^{6}), B_l were obtained for $l \leqslant 5$, but, using in addition to the values of the boundary function the values of its derivatives, it is possible to obtain B_6 with nonnegative coefficients. Operators of the type D_{x_j} are used in practice in (^{10,11}), but without an estimate of the error of the solution. The results obtained carry over to the Dirichlet problem in a multiply connected domain bounded by a polygonal line and may be used for other grids, for example, a variable one.

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Note: Figure translations are in progress. See original paper for figures.