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Abstract

Full Text

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INVESTIGATION OF A DIFFERENTIAL EQUATION OF STURM–LIOUVILLE TYPE

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1. An equation of Sturm–Liouville type

$$z^{\mathrm{IV}}(x) + [\lambda - q(x)]z(x) = 0, \qquad a \le x \le b, \tag{1}$$

is considered under the assumptions: 1) continuity of the function $q(x)$ on (a, b) and 2) existence of the finite limits $\lim_{x\to a+0} q(x)$, $\lim_{x\to b-0} q(x)$ (similarly to $(*)$).

A solution of equation (1) satisfying certain boundary conditions is an entire function of λ for all $x \in [a, b]$. To each eigenvalue λ there correspond four linearly independent eigenfunctions of equation (1): $\varphi(x, \lambda)$, $\psi(x, \lambda)$, $\vartheta(x, \lambda)$, $\eta(x, \lambda)$. Their Wronskian $W[\varphi, \psi, \vartheta, \eta]$ does not depend on x and is an entire function of λ. We shall consider the third-order minors of the Wronskian, denoting by $A[\varphi^{(i)}]$ the minor complementary to the element $\varphi^{(i)}(x, \lambda)$ in the Wronskian, $i = 0, 1, 2, 3$. These minors, which are functions of x and λ, will be differentiated with respect to x.

Lemma 1.

$$\{A[\varphi^{(i)}]\}^{(j)} = A[\varphi^{(i-j)}], \quad i = 1, 2, 3; \quad j \le i, \qquad \{A[\varphi]\}' = [q(x) - \lambda]A[\varphi'''].$$

Lemma 2. $A[\varphi'''](x, \lambda)$ is an eigenfunction of equation (1), linearly independent of $\varphi(x, \lambda)$, corresponding to the eigenvalue λ.

Lemma 3. If the boundary conditions

$$\varphi^{(k+2)}(x, \lambda) = -p(\lambda)\varphi^{(k)}(x, \lambda), \qquad \psi^{(k+2)}(x, \lambda) = -p(\lambda)\psi^{(k)}(x, \lambda),$$

$$\vartheta^{(k+2)}(x, \lambda) = p(\lambda)\vartheta^{(k)}(x, \lambda), \qquad \eta^{(k+2)}(x, \lambda) = p(\lambda)\eta^{(k)}(x, \lambda)$$

are satisfied at the point $x = a$ (or at the point $x = b$), $k = 0, 1$, then the determinants

$$\begin{vmatrix} \vartheta'(x,\lambda) & \eta'(x,\lambda) \\ \vartheta''(x,\lambda) & \eta''(x,\lambda) \end{vmatrix}, \qquad \begin{vmatrix} \varphi'(x,\lambda) & \psi'(x,\lambda) \\ \varphi''(x,\lambda) & \psi''(x,\lambda) \end{vmatrix}$$

do not depend on x,

$$A[\varphi'''] = 2 \begin{vmatrix} \vartheta'(x,\lambda) & \eta'(x,\lambda) \\ \vartheta''(x,\lambda) & \eta''(x,\lambda) \end{vmatrix} \psi(x,\lambda),$$

$$A[\psi'''] = 2 \begin{vmatrix} \vartheta'(x,\lambda) & \eta'(x,\lambda) \\ \vartheta''(x,\lambda) & \eta''(x,\lambda) \end{vmatrix} \varphi(x,\lambda),$$

$$A[\vartheta'''] = 2 \begin{vmatrix} \varphi'(x,\lambda) & \psi'(x,\lambda) \\ \varphi''(x,\lambda) & \psi''(x,\lambda) \end{vmatrix} \eta(x,\lambda),$$

$$A[\eta'''] = 2 \begin{vmatrix} \varphi'(x,\lambda) & \psi'(x,\lambda) \\ \varphi''(x,\lambda) & \psi''(x,\lambda) \end{vmatrix} \vartheta(x,\lambda),$$

$$W[\varphi,\psi,\vartheta,\eta] = -4 \begin{vmatrix} \varphi'(x,\lambda) & \psi'(x,\lambda) \\ \varphi''(x,\lambda) & \psi''(x,\lambda) \end{vmatrix} \cdot \begin{vmatrix} \vartheta'(x,\lambda) & \eta'(x,\lambda) \\ \vartheta''(x,\lambda) & \eta''(x,\lambda) \end{vmatrix}.$$

If the boundary values of the functions $\varphi(x,\lambda)$, $\vartheta(x,\lambda)$ at $x = a$ and the boundary values of the functions $\psi(x,\lambda)$, $\eta(x,\lambda)$ at $x = b$ do not depend on λ, then the determinants

$$\begin{vmatrix} \varphi' & \psi' \\ \varphi'' & \psi'' \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} \vartheta' & \eta' \\ \vartheta'' & \eta'' \end{vmatrix}$$

have real and simple zeros.

The function

$$u(x,\lambda) = \frac{A[\varphi''']}{W} \int_a^x \varphi(y,\lambda) f(y)\, dy + \frac{A[\psi''']}{W} \int_x^b \psi(y,\lambda) f(y)\, dy +$$

$$+ \frac{A[\vartheta''']}{W} \int_a^x \vartheta(y,\lambda) f(y)\, dy + \frac{A[\eta''']}{W} \int_x^b \eta(y,\lambda) f(y)\, dy$$

is a solution of the equation

$$x^{\mathrm{IV}}(x) + [\lambda - q(x)] x(x) = f(x), \qquad \text{where } f(x) \in L[a,b].$$

Under the assumptions made in Lemma 3,

$$x(x,\lambda) = -\frac{\psi(x,\lambda)}{2\begin{vmatrix}\varphi' & \psi' \\ \varphi'' & \psi''\end{vmatrix}} \int_a^x \varphi(y,\lambda) f(y)\,dy - \frac{\varphi(x,\lambda)}{2\begin{vmatrix}\varphi' & \psi' \\ \varphi'' & \psi''\end{vmatrix}} \int_x^b \psi(y,\lambda) f(y)\,dy$$
$$- \frac{\eta(x,\lambda)}{2\begin{vmatrix}\vartheta' & \eta' \\ \vartheta'' & \eta''\end{vmatrix}} \int_a^x \vartheta(y,\lambda) f(y)\,dy - \frac{\vartheta(x,\lambda)}{2\begin{vmatrix}\vartheta' & \eta' \\ \vartheta'' & \eta''\end{vmatrix}} \int_x^b \eta(y,\lambda) f(y)\,dy.$$

$$\sum_{\lambda_k} \operatorname{Res} x(x,\lambda)$$

(λ_k are simple poles at which at least one of the relations $\varphi(x,\lambda_k) = c_k\psi(x,\lambda_k)$, $\vartheta(x,\lambda_k) = d_k\eta(x,\lambda_k)$ is fulfilled) represents the expansion of the function $f(x)$ in eigenfunctions of equation (1).

2. Let $\Phi(x,\lambda)$, $\Psi(x,\lambda)$, $\Theta(x,\lambda)$, $H(x,\lambda)$ be linearly independent eigenfunctions of equation (1) for $q(x) \equiv 0$. They may be chosen so that $A[\Phi'''] = A[\Phi''] = A'[\Phi'] = 0$, $A[\Phi] = -s^6$ at some point of the interval $[a,b]$. For example:

$$\Phi(y,\lambda) = \frac{1}{\sqrt{2}}\,\mathrm{ch}[s(y-x)],$$

$$\Theta(y,\lambda) = \frac{1}{\sqrt{2}}\{\mathrm{ch}[s(y-x)] - \cos[s(y-x)]\},$$

$$\Psi(y,\lambda) = \frac{1}{\sqrt{2}}\,\mathrm{sh}[s(y-x)],$$

$$H(y,\lambda) = \frac{1}{\sqrt{2}}\{\mathrm{sh}[s(y-x)] - \sin[s(y-x)]\}, \qquad s = \sqrt[4]{\lambda}.$$

By fourfold integration by parts in

$$\int_a^x A[\Phi''']\,\varphi^{\mathrm{IV}}(y)\,dy$$

one establishes the relation

$$\int_a^x A[\Phi''']\,[\varphi^{\mathrm{IV}}(y) + \lambda\varphi(y)]\,dy = W[\varphi,\Psi,\Theta,H]\Big|_x^a,$$

which may be rewritten in the form

$$\int_a^x A[\Phi'''](y,\lambda)q(y)\varphi(y,\lambda)\,dy =$$

$$= W[\varphi(y,\lambda),\Psi(y,\lambda),\Theta(y,\lambda),H(y,\lambda)]\Big|_{y=a} + s^6\varphi(x,\lambda);$$

expanding the Wronskian along the first column, we obtain:

$$\begin{aligned}\varphi(x,\lambda) =&\varphi(a,\lambda)\frac{\mathrm{ch}[s(a-x)]+\cos[s(a-x)]}{2} - \varphi'(a,\lambda)\frac{\mathrm{sh}[s(a-x)]+\sin[s(a-x)]}{2s}\\ &+\varphi''(a,\lambda)\frac{\mathrm{ch}[s(a-x)]-\cos[s(a-x)]}{2s^2} - \varphi'''(a,\lambda)\frac{\mathrm{sh}[s(a-x)]-\sin[s(a-x)]}{2s^3}\\ &+\frac{1}{2s^3}\int_a^x\{\sin[s(y-x)]-\mathrm{sh}[s(y-x)]\}q(y)\varphi(y,\lambda)\,dy.\end{aligned} \tag{2}$$

This integral equation, asymptotic in s, is the main one in the subsequent considerations. Analogous formulas are easily obtained for $\psi(x,\lambda)$, $\vartheta(x,\lambda)$, $\eta(x,\lambda)$.

Theorem 1. *The solution of the boundary-value problem for equation (1) is an eigenfunction of the integral equation (2).*

Analogous asymptotic formulas can be obtained for solutions of the boundary-value problem of the equation $z^{(2n)}(x) + [\lambda - q(x)]z(x) = 0$, n arbitrary. The larger n is, the smaller asymptotically is the magnitude of the integral term in comparison with the nonintegral one, which is the solution of the boundary-value problem for the equation $z^{(2n)}(x) + \lambda z(x) = 0$, i.e., the smaller is the influence of the function $q(x)$.

If, in formula (2), we restrict ourselves to the first two terms (and proceed in the same way in the asymptotic expressions for $\psi(x,\lambda)$, $\vartheta(x,\lambda)$, $\eta(x,\lambda)$, considering the boundary values $\vartheta(x,\lambda)$ given at $x = a$ and the boundary values $\psi(x,\lambda)$, $\eta(x,\lambda)$ given at $x = b$), and also require

$$1)\quad \begin{vmatrix}\varphi(a) & \vartheta(a)\\ \varphi'(a) & \vartheta'(a)\end{vmatrix}\cdot\begin{vmatrix}\psi(b) & \eta(b)\\ \psi'(b) & \eta'(b)\end{vmatrix} \neq 0,$$

$$2)\quad \begin{vmatrix}\psi(b) & \eta(b)\\ \psi'(b) & \eta'(b)\end{vmatrix}\cdot\begin{vmatrix}\varphi'(a) & \vartheta'(a)\\ \varphi''(a) & \vartheta''(a)\end{vmatrix} - \begin{vmatrix}\varphi(a) & \vartheta(a)\\ \varphi'(a) & \vartheta'(a)\end{vmatrix}\cdot\begin{vmatrix}\psi'(b) & \eta'(b)\\ \psi''(b) & \eta''(b)\end{vmatrix} = 0,$$

then we obtain the asymptotic formula for the Wronskian

$$W[\varphi(x,\lambda),\psi(x,\lambda),\vartheta(x,\lambda),\eta(x,\lambda)] =$$

$$= \lambda \left\{ \begin{vmatrix} \varphi(a) & \vartheta(a) \\ \varphi'(a) & \vartheta'(a) \end{vmatrix} \cdot \begin{vmatrix} \psi(b) & \eta(b) \\ \psi'(b) & \eta'(b) \end{vmatrix} \operatorname{ch}[s(a-b)] \cos[s(a-b)] + O\left(\frac{e^{|s|}}{s^2}\right) \right\}.$$

Theorem 2. Under the assumptions made, the zeros of the Wronskian form two sequences

$$s_k = \frac{k\pi}{a-b} + O(k^{-2}), \qquad t_k = \frac{k\pi i}{a-b} + O(k^{-2}), \qquad k = 0, \ \pm 1, \ \pm 2, \ldots$$

For both sequences, $\lambda_k = \left(\frac{k\pi}{a-b}\right)^4 + O(k)$ are real.

Theorem 2 is analogous to the theorem on the growth of the eigenvalues of integral equations with a kernel of Green' s-function type (these eigenvalues are zeros of a certain entire function) (2).

3. Under the additional assumptions

$$\varphi^{(k)}(a, \lambda) = -\frac{1}{s^2}\varphi^{(k+2)}(a, \lambda), \qquad \varphi^{(k)}(a, \lambda) = -\frac{1}{s^2}\varphi^{(k+2)}(a, \lambda),$$

$$\vartheta^{(k)}(a, \lambda) = \frac{1}{s^2}\vartheta^{(k+2)}(a, \lambda), \qquad \eta^{(k)}(a, \lambda) = \frac{1}{s^2}\eta^{(k+2)}(a, \lambda), \qquad k = 0, 1,$$

$$\frac{\psi'(b, \lambda)}{\psi(b, \lambda)} = \frac{\varphi'(a, \lambda)}{\varphi(a, \lambda)}, \qquad \frac{\eta'(b, \lambda)}{\eta(b, \lambda)} = \frac{\vartheta'(a, \lambda)}{\vartheta(a, \lambda)}$$

we may assert the following:

β) For $s = s_k, \ k = 0, \ \pm 1, \ \pm 2, \ldots$, there is the asymptotic proportionality

$$\eta(x, \lambda_k) = (-1)^k \frac{\eta(b, \lambda_k)}{\vartheta(a, \lambda_k)} \vartheta(x, \lambda_k) + O\left(\frac{e^{|s|}}{s^3}\right),$$

and, moreover, no asymptotic proportionality

$$\varphi(x, \lambda_k) = c\psi(x, \lambda_k) + O(e^{|s|}/s^3)$$

is possible for any c.

2) For $s = t_k, \ k = 0, \pm 1, \pm 2 \ldots$, there is the asymptotic proportionality

$$\psi(x,\lambda_k)=(-1)^k\frac{\psi(b,\lambda_k)}{\varphi(a,\lambda_k)}\varphi(x,\lambda_k)+O\left(\frac{e^{|s|}}{s^3}\right),$$

and, moreover, no asymptotic proportionality

$$\vartheta(x,\lambda_k)=c\eta(x,\lambda_k)+O(e^{|s|}/s^3)$$

is possible for any c.

In this case we obtain the asymptotic formulas

$$\frac{A[\varphi''']}{W}=\frac{\cos[s(x-b)]}{2s^3\varphi(a)\sin[s(a-b)]}+O\left(\frac{e^{|s|}}{s^5}\right),$$

$$\frac{A[\psi''']}{W}=\frac{\cos[s(a-x)]}{2s^3\psi(b)\sin[s(a-b)]}+O\left(\frac{e^{|s|}}{s^5}\right),$$

$$\frac{A[\vartheta''']}{W}=\frac{\mathrm{ch}[s(x-b)]}{2s^3\vartheta(a)\,\mathrm{sh}[s(a-b)]}+O\left(\frac{e^{|s|}}{s^5}\right),$$

$$\frac{A[\eta''']}{W}=\frac{\mathrm{ch}[s(a-x)]}{2s^3\eta(b)\,\mathrm{sh}[s(a-b)]}+O\left(\frac{e^{|s|}}{s^5}\right).$$

$$\begin{aligned}\varkappa(x,\lambda)=&\frac{1}{2s^3}\frac{\cos[s(x-b)]}{\sin[s(a-b)]}\int_a^x\cos[s(a-y)]f(y)\,dy+\\&+\frac{1}{2s^3}\frac{\cos[s(a-x)]}{\sin[s(a-b)]}\int_x^b\cos[s(y-b)]f(y)\,dy+\\&+\frac{1}{2s^3}\frac{\mathrm{ch}[s(x-b)]}{\mathrm{sh}[s(a-b)]}\int_a^x\mathrm{ch}[s(a-y)]f(y)\,dy+\\&+\frac{1}{2s^3}\frac{\mathrm{ch}[s(a-x)]}{\mathrm{sh}[s(a-b)]}\int_a^b\mathrm{ch}[s(y-b)]f(y)\,dy+O\left(\frac{e^{|s|}}{s^5}\right),\end{aligned}$$

$$\sum_{s_k,t_k}\mathrm{Res}\,\varkappa(x,\lambda)=\lim_{k\to\infty}\frac{1}{2\pi i}\int_{\Gamma_k}\varkappa(x,\lambda)\,d\lambda.$$

The contour of integration Γ_k is obtained under the mapping $\lambda=s^4$ of the contour S_k of the s-plane; S_k is the boundary of a square with sides parallel to the coordinate axes, with side lengths $(2k+1)\pi$, the square being symmetric with respect to the coordinate axes.

$$\sum_{s_k,t_k} \operatorname{Res} \varkappa(x,\lambda) = \sum_{k=0}^{\infty} \frac{4}{(a-b)} \Bigg\{ \int_a^x \frac{\cos[s_k(x-b)]\cos[s_k(a-y)]}{\cos[s_k(a-b)]} f(y)\,dy + \\ + \int_x^b \frac{\cos[s_k(a-x)]\cos[s_k(y-b)]}{\cos[s_k(a-b)]} f(y)\,dy + \\ + \int_a^x \frac{\operatorname{ch}[t_k(x-b)]\operatorname{ch}[t_k(a-y)]}{\operatorname{ch}[t_k(a-b)]} f(y)\,dy + \\ + \int_a^b \frac{\operatorname{ch}[t_k(a-x)]\operatorname{ch}[t_k(b-y)]}{\operatorname{ch}[t_k(a-b)]} f(y)\,dy + O\left(\frac{e^{|s_k|}}{s_k}\right) \Bigg\}. \tag{3}$$

Theorem 3. The asymptotic formula (3) for the expansion of an arbitrary function $f(x) \in L[a,b]$ in the eigenfunctions of equation (1), which is an analogue of the trigonometric Fourier series, is valid. If $f(x)$ has bounded variation, then the expansion converges to

$$\frac{1}{2}\,[f(x+0)+f(x-0)].$$

4. One may consider, analogously to what was set forth above, the equation

$$z^{(2n)}(x) + [\lambda - q(x)]z(x) = 0, \qquad a \le x \le b.$$

The preservation of properties of the Wronskian minors is verified; an asymptotic formula analogous to (2) is constructed for solutions of the boundary-value problem; $\varkappa(x,\lambda)$ is constructed as the solution of the equation

$$z^{(2n)}(x) + [\lambda - q(x)]z(x) = f(x)$$

for any function $f(x) \in L[a,b]$. The zeros of the Wronskian $W[\lambda]$ grow, $\lambda_k = O(k^{2n})$, and are real.

Under additional assumptions on the boundary values of the eigenfunctions of equation (4), the expansion of an arbitrary $f(x) \in L[a,b]$ in the eigenfunctions into a series analogous to the trigonometric Fourier series is constructed.

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CITED LITERATURE

1. E. C. Titchmarsh, *Eigenfunction Expansions Associated with Second-Order Differential Equations*, 1, Moscow, 1960.

2. M. Kh. Zakhar-Itkin, DAN, **158**, No. 5 (1964).

Note: Figure translations are in progress. See original paper for figures.