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Abstract

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POWER EXPANSION FOR ONE-DIMENSIONAL BOUNDARY-VALUE PROBLEMS

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An approximate solution of a one-dimensional boundary-value problem reduces to the choice of an approximating solution function that exactly satisfies the boundary conditions at the ends of a finite interval from 0 to l. In addition to the boundary conditions, one should, as far as possible, exactly satisfy the differential equation

$$F(y, s) = 0 \tag{1}$$

of the boundary-value problem.

We shall seek the solution in the form of a polynomial of order n

$$y = \sum_0^n a_k \frac{s^k}{k!}, \tag{2}$$

whose coefficients a_k are the values of the derivatives of this polynomial at the point $s = 0$. Taking into account the natural equality of the endpoints of the interval of length l, it proves expedient in the solution (2) to represent both ends of the interval on an equal footing. Grouping the terms of the original polynomial, one can obtain a polynomial of the form $y = \sum_0^n b_k \frac{(s-l)^k}{k!}$, where the coefficients b_k are connected by a system of $n+1$ linear equations with the original coefficients a_k. Thanks to this, one can eliminate the higher coefficients from both forms of the solution and obtain a polynomial that is simultaneously expressed in terms of the values of the derivatives at both endpoints of the interval.

In such polynomials, independently of their order, each preceding polynomial is contained in the subsequent ones. This makes it possible to generalize the results obtained to the case of infinite degree and to obtain a power expansion of a function in boundary values in a series of the form

$$
\begin{aligned}
f(s) = \frac{1}{2}[f(0)+f(l)] + \sum_{1}^{\infty} \frac{1}{2}\left[f^{(n)}(0)+(-1)^n f^{(n)}(l)\right] l^n \times \\
\times \sum_{m=n}^{\infty} A_{nm}\left(1-\frac{s}{l}\right)^m \frac{s^m}{l^m}\, 4^m + \sum_{0}^{\infty} \frac{1}{2}\left[f^{(n)}(0)-(-1)^n f^{(n)}(l)\right] l^n \times \\
\times \sum_{m=n}^{\infty} B_{nm}\left(1-\frac{s}{l}\right)^m \frac{s^m}{l^m}\left(1-2\frac{s}{l}\right) 4^m,
\end{aligned}
\tag{3}
$$

where

$$
A_{1m} = \frac{(-1)}{2^{m+1} m!} \prod_{0}^{m-1}(2k-1), \qquad A_{2m} = \frac{(-1)}{2^{m+2} m!} \prod_{0}^{m-1}(2k-1),
$$

$$
A_{3m} = \frac{(-1)}{3!\, 2^{m+3} m!} \cdot \frac{m-2}{2m-3} \prod_{0}^{m-1}(2k-1); \ldots
$$

$$
B_{0m} = \frac{(-1)}{2^m m!} \prod_{0}^{m}(2k-1), \qquad B_{1m} = \frac{(-1)}{2^{m+1} m!} \prod_{0}^{m}(2k-1) \ldots .
$$

coefficients of the expansion, recurrence formulas for which can be obtained by comparing the results of differentiating such an expansion with the direct representation, in the form (3), of the derivative of the same function.

From the new power expansion there follows a quadrature formula which unifies analogous formulas of C. Lanczos (3).* Lanczos obtained a convergence condition for a quadrature formula using boundary values, which reduces to the regularity of the integrand in a certain region of the complex plane containing the interval from 0 to l. He showed that the radius of convergence exceeds this interval by $0.207\,l$ on each side. The results obtained by Lanczos also apply to the new power expansion, owing to which convergence of the analytic approximation (3) in the boundary-value problem on the interval from 0 to l will be ensured. In view of this, polynomial (2) will be the main part of the analytic approximation, since it is the initial part of expansion (3).

The boundary conditions of the boundary-value problem, whose number is equal to the order p of the differential equation (1), are expressed in the approximate solution through the constants a_k and b_k. Since the boundary conditions are expressed through the values of the derivatives of the desired function at the ends of the interval, it seems possible to satisfy them in the approximate solution exactly.

With the aid of the same constants it is possible to satisfy exactly the differential equation (1) at both ends of the interval, using for this purpose two algebraic equations of the form

$$F(0) = 0, \qquad F(l) = 0. \tag{4}$$

If, with the aid of the same constants, the derivatives of the differential equation are also satisfied in order to determine all $n+1-p$ constants, then one obtains a generalization of the well-known method of power series for solving boundary-value problems, with simultaneous satisfaction of the derivatives of equation (1) at both ends of the interval. The formal application of the method of power series enabled the author ([1],[2]) to obtain a solution of nonlinear boundary-value problems with nonhomogeneous boundary conditions. However, such a method turns out not to be the most effective.

For expressing the error function which is formed after substituting solution (2) into equation (1), it is advisable to use the power expansion in the form (3), since after this it becomes possible to pass to an expansion of the error in the mean over the interval in orthogonal polynomials, in particular in the optimal Chebyshev polynomials. Thus, the prerequisites are created for applying P. L. Chebyshev' s method of least deviation by his method of corrections ([4]) and through boundary values simultaneously.

As a result, after complete elimination of the error at the ends of the interval with the aid of expressions (4), the most effective minimization of the error inside the interval proves to be achieved by successively setting equal to zero the coefficients at the initial Chebyshev polynomials with the aid of equations of the form

$$\frac{1}{2}\sum_{k=0}^{t} C_{kt}\left[F^{(k)}(0) + (-1)^k F^{(k)}(l)\right] l^k = 0, \tag{5}$$

$$\frac{1}{2}\sum_{k=0}^{t} D_{kt}\left[F^{(k)}(0) - (-1)^k F^{(k)}(l)\right] l^k = 0, \tag{6}$$

which correspond respectively to the zero-th and first Chebyshev polynomials.

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* These formulas became known to the author after the completion of the present work. Our work was reported at the conference on the problem of elastic vibrations in Riga on June 22, 1961.

The separate and exact fulfillment of these conditions may be interpreted as the orthogonalization of the error and, consequently, as the implementation of Galerkin' s method by non-integral means. However, there is no need for the exact satisfaction of conditions of the form (5) and (6), if one takes into account that expansion (3) singles out from these conditions the principal initial part.

Indeed, the successive values of the quantities C_{kt} and D_{kt}, presented in Table 1, for any order t of the error differ numerically by almost an order of magnitude, which makes it possible to simplify the orthogonalization conditions within the required accuracy.

Table 1

Coefficient	k	$t = 3$	$t = 5$	$t = 7$	$t = 9$	$t = 11$	$t = \infty$
C_{kt}	0	1	1	1	1	1	1
C_{kt}	1	$1,250 \cdot 10^{-1}$	$1,484 \cdot 10^{-1}$	$1,582 \cdot 10^{-1}$	$1,638 \cdot 10^{-1}$	$1,660 \cdot 10^{-1}$	$1,81 \cdot 10^{-1}$
C_{kt}	2		$1,175 \cdot 10^{-2}$	$1,660 \cdot 10^{-2}$	$1,942 \cdot 10^{-2}$	$2,052 \cdot 10^{-2}$	$2,80 \cdot 10^{-2}$
C_{kt}	3			$0,82 \cdot 10^{-3}$	$1,33 \cdot 10^{-3}$	$1,68 \cdot 10^{-3}$	$3,37 \cdot 10^{-3}$
C_{kt}	4				$0,43 \cdot 10^{-4}$	$0,72 \cdot 10^{-4}$	$2,13 \cdot 10^{-4}$
D_{kt}	0	1,125	1,172	1,197	1,213	–	1,270
D_{kt}	1	$0,625 \cdot 10^{-1}$	$0,859 \cdot 10^{-1}$	$0,981 \cdot 10^{-1}$	$1,075 \cdot 10^{-1}$	–	$1,35 \cdot 10^{-1}$
D_{kt}	2		$0,391 \cdot 10^{-2}$	$0,637 \cdot 10^{-2}$	$0,798 \cdot 10^{-2}$	–	$2,05 \cdot 10^{-2}$
D_{kt}	3			$0,204 \cdot 10^{-3}$	$0,338 \cdot 10^{-3}$	–	$1,38 \cdot 10^{-3}$

In turn, equations (4) are naturally combined with approximate conditions for the orthogonalization of the error, simplifying and mutually complementing one another. Thus, by combining a number of approximate methods, one obtains a single method that is more effective than each method separately. Already the first approximation on the basis of equations (4) and one equation (5) reveals, as a rule, the general character of the process. Thus, in the problem of the vibration of a string, the first approximation gives an error in the frequency of the fundamental tone of about 0,5%.

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Note: Figure translations are in progress. See original paper for figures.