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Abstract

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ON THE SOLVABILITY OF BOUNDARY-VALUE PROBLEMS FOR QUASILINEAR ELLIPTIC SYSTEMS

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Let Ω be an n-dimensional domain with sufficiently smooth boundary Γ. Consider the operator

$$Au(x) = \sum_{|\alpha|=|\beta|=m} (-1)^m D^\alpha \left[a_{\alpha\beta}(x) D^\beta u(x)\right],$$

where $a_{\alpha\beta}$ are real, symmetric, sufficiently smooth matrices of order N^2, satisfying the condition $a_{\alpha\beta} = a_{\beta\alpha}$. Suppose that for any function $u(x) \in C_0^\infty(\Omega)$, finite, infinitely differentiable, with compact support in Ω, the inequality

$$(Au, u)_{L_2} \geq \mu^2 \|u\|^2_{\overset{0}{W}{}_2^{(m)}}, \quad \mu > 0. \tag{1}$$

is fulfilled.

Then the operator A can be extended in a unique way to the space $\overset{0}{W}{}_2^{(m)}(\Omega)$ to a self-adjoint positive definite operator, and inequality (1) remains valid for the extension of the operator A and all $u(x) \in \overset{0}{W}{}_2^{(m)}(\Omega)$ [1]. Assuming such an extension has been carried out, we consider the problem of the existence of solutions of the equation

$$Au(x) = f\left[x, u(x), D^\gamma u(x)\right], \quad |\gamma| \leq m, \tag{2}$$

belonging to the space $\overset{0}{W}{}_2^{(m)}(\Omega)$. In the present work it is proved that, under the fulfillment of certain conditions, the boundary-value problem has at least one solution. In the case when $f(x, 0, 0) \neq 0$, i.e., when the function $u(x) \equiv 0$ is not a solution of the boundary-value problem, conditions are given for the existence of at least one solution distinct from the identical zero.

1. Suppose that the function $f(x, u, \zeta_\gamma)$ is continuous in the aggregate of the variables $x \in \overline{\Omega}$, $|u|$, $|\zeta_\gamma| < \infty$, and satisfies the inequalities

$$|f(x,u,\zeta_\gamma)| \leq M(x,u,\zeta_\delta)\left[1+\sum_{k=k_0+1}^{m}\sum_{|\gamma|=1}|\zeta_\gamma|^{p_k}\right]^{1/p}; \tag{3}$$

$$uf(x,u,\zeta_\gamma) \leq a(x)+\sigma_0|u|^2+\sum_{k=1}^{m}\sigma_k\sum_{|\gamma|=k}|\zeta|^2, \tag{4}$$

where $M(x,u,\zeta_\delta)$ is continuous in the aggregate of the variables $x \in \overline{\Omega}$, $|u|$, $|\zeta_\delta| < \infty$, $|\delta| \leq k_0 = m-[n/2]$, $p_k = 2+\dfrac{4(2m-k)}{n-2(2m-k)}$, $p > \dfrac{2n}{n+2m}$, $a(x) \in L(\Omega)$, and σ_k are nonnegative constants that will be determined below.

Theorem 1. *If conditions (3) and (4) are fulfilled, then equation (2) has at least one solution* $u(x) \in \overset{0}{W}{}_2^{(m)}(\Omega)$.

Proof. Let $u(x) \in \overset{0}{W}{}_2^{(m)}(\Omega)$ be a solution of equation (2). Then, by virtue of (1),

$$\mu^2\|u\|^2_{\overset{0}{W}{}_2^{(m)}}(Au,u)_{L_2} = \left(f\left[x,u(x),D^\gamma u(x)\right],u(x)\right)_{L_2}.$$

Hence, by virtue of (4), it follows that

$$\mu^2\|u\|_{\overset{0}{W}{}_2^{(m)}} \leq a_0+\sum_{k=0}^{m}c_k\sigma_k\|u\|^2_{\overset{0}{W}{}_2^{(m)}},$$

where c_k are the norms of the embedding operators of $\overset{0}{W}{}_2^{(m)}(\Omega)$ into $W_2^{(k)}(\Omega)$, $0 \leq k \leq m$, $a_0 = \int_\Omega a(x)\,dx$.

Assuming that $\sum_{k=0}^{m}c_k\sigma_k < \mu^2$, we obtain from this that

$$\|u(x)\|_{\overset{0}{W}{}_2^{(m)}} \leq \left[a_0\Big/\left(\mu^2-\sum_{k=0}^{m}c_k\sigma_k\right)\right]^{1/2}. \tag{5}$$

The operator A establishes a one-to-one and continuous mapping of the space $\overset{0}{W}{}_2^{(m)}(\Omega)$ onto the space $\overset{0}{W}{}_2^{(-m)}(\Omega) = (\overset{0}{W}{}_2^{(m)})^*$; moreover the operator A^{-1} admits the factorization $A^{-1} = TT^*$, where T is a continuous operator acting from $L_2(\Omega)$ to $\overset{0}{W}{}_2^{(m)}(\Omega)$, and T^* is a continuous operator acting from $\overset{0}{W}{}_2^{(-m)}(\Omega)$ to $L_2(\Omega)$ [2–4].

Since $p > 2n/(n+2m)$, we have $L_p(\Omega) \subset \overset{0}{W}{}_2^{(-m)}(\Omega)$. Consequently, the operator T^* is completely continuous as an operator acting from $L_p(\Omega)$, $p > 2n/(n+2m)$, to $L_2(\Omega)$.

Consider in $L_2(\Omega)$ the operator equation

$$v(x) = T^* f[x, Tv(x), D^\gamma Tv(x)]. \tag{6}$$

It is easy to see that, if the function $v(x) \in L_2(\Omega)$ is a solution of equation (6), then the function $u(x) = Tv(x)$ belongs to $\overset{0}{W}{}_2^{(m)}(\Omega)$ and satisfies equation (2).

To prove the solvability of equation (6), we use the theorem (see ([5])):

If a completely continuous operator B, acting in a Hilbert space, satisfies the inequality

$$(Bu, u) < (u, u), \qquad \|u\| = r,$$

then the equation $u = Bu$ has at least one solution u_0, and $\|u_0\| < r$.

Let $\|v(x)\|_{L_2} = r$, $r > \left[a_0 / \left(\mu^2 - \sum_{k=1}^m c_k \sigma_k\right)\right]^{1/2}$. Then, putting

$$Bv(x) = T^* f[x, Tv(x), D^\gamma Tv(x)],$$

we shall have, by virtue of (4),

$$\begin{aligned}(Bv, v)_{L_2} &= \left(T^* f[x, Tv(x), D^\gamma Tv(x)], v(x)\right)_{L_2} \\ &= \int_\Omega f[x, Tv(x), D^\gamma Tv(x)]\, Tv(x)\, dx \\ &\leq \|T\|^2 \left(a_0 + \sum_{k=0}^m c_k \sigma_k \|v\|_{L_2}^2\right),\end{aligned}$$

where $\|T\|$ is the norm of the operator T, acting from $L_2(\Omega)$ to $\overset{0}{W}{}_2^{(m)}(\Omega)$. It is easy to see that $\|T\| < 1/\mu$. Consequently,

$$(Bv, v)_{L_2} < (v, v)_{L_2}, \qquad \|v\|_{L_2} = r.$$

Thus, equation (6), and consequently also the boundary-value problem, have at least one solution.

We note that all the arguments remain valid if, instead of the space $\mathring{W}_2^{(m)}(\Omega)$, one takes any closed subspace $V \subset \mathring{W}_2^{(m)}(\Omega)$ containing $C_0^\infty(\Omega)$. It is easy to show that the solutions of the boundary-value problems are continuously differentiable $2m-1$ times.

2. Let now $f(x,0,0) \equiv 0$. Then equation (2) has in $\overset{\circ}{W}{}_2^{(m)}(\Omega)$ the trivial solution $u(x) \equiv 0$. The following theorem on the existence of solutions of the boundary-value problem different from the trivial one holds.

Theorem 2. *Let the conditions of Theorem 1 be satisfied. Suppose, in addition, that:*

a) $f(x,0,0) \equiv 0$;

b) *the function $f(x,u,\xi_\gamma)$ has partial derivatives with respect to the variables $u=(u_1,\dots,u_N)$ and all variables ξ_γ, $|\gamma| \leqslant m$, continuous in some neighborhood of the point $u=0$, $\xi_\gamma = 0$;*

c) *the matrix $f_u(x,0,0) = (f_{iu_j}(x,0,0))$ does not depend on x, and the matrices $f_{\xi_\gamma}(x,0,0) \equiv 0$;*

d) *there exists an independent, positive definite, completely continuous operator Q, acting in $L_2(\Omega)$, such that*

$$\mu_1 Qv(x) \leqslant T^* f_u(x,0,0) Tv(x) \leqslant \mu_2 Qv(x), \qquad v(x) \in L_2(\Omega), \tag{7}$$

where

$$\sqrt{\mu_2^2 - \mu_1^2} < \min_i \left| \mu_2 - \frac{1}{\lambda_i} \right|,$$

where λ are the eigenvalues of the operator Q;

e) *the number μ_2 belongs to one of the intervals*

$$\left(\frac{1}{\lambda_{2i-1}}, \frac{1}{\lambda_{2i}} \right), \qquad i = 1, 2, \dots .$$

Then equation (2) has at least one solution $u(x) \in \overset{\circ}{W}{}_2^{(m)}(\Omega)$ different from the trivial one.

We give the scheme of the proof of Theorem 2.

Under the conditions of Theorem 1, the rotation of the vector field $I-B$ on the sphere $\|v\| = r$,

$$r > \left[a_0 / \left(\mu^2 - \sum_{k=0}^{m} c_k \sigma_k \right) \right]^{1/2},$$

is equal to 1. The operator $B_1 v(x) = T^* f_u(x,0,0) T v(x)$ is the Fréchet derivative of the operator

$$Bv(x) = Tf[x, Tv(x), D^\gamma Tv(x)]$$

at the point $v(x) \equiv 0$.

From inequalities (7) and (8) it is easy to derive the homotopy of the vector fields $I - B_1$ and $I - \mu Q$ ($\mu_1 \leqslant \mu \leqslant \mu_2$) on spheres with center at the origin. From the homotopy of vector fields and the theorem of Leray and Schauder [(6)], it follows that the index of the zero solution is equal to $(-1)^\beta$, where β is the sum of the multiplicities of the characteristic numbers of the operator Q lying in the interval $(0, \mu)$.

Since, by d), β is odd, the index of the zero solution is equal to -1, while the rotation of the vector field $I - B$ on the sphere $\|v(x)\| = r$ is equal to 1; hence there exists at least one solution $v(x) \neq 0$ of equation (6), with $\|v(x)\|_{L_2} < r$. Then the function $u(x) = Tv(x)$ will be a nonzero solution of the boundary-value problem.

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CITED LITERATURE

1. K. Fridrichs, *Math. Ann.*, **109**, 4–5 (1934).
2. M. I. Vishik, *Tr. Mosk. matem. obshch.*, **12**, 125 (1963).
3. Yu. M. Berezanskii, *UMN*, **13**, no. 1 (109) (1963).
4. M. A. Krasnosel' skii, S. G. Krein, *Tr. seminara po funktsional' n. analizu*, Voronezh, no. 5 (1957).
5. M. A. Krasnosel' skii, *Izv. AN SSSR, ser. matem.*, **20**, no. 2, 241 (1956).
6. J. Leray, J. Schauder, *UMN*, **1**, no. 3–4, 71 (1946).

Note: Figure translations are in progress. See original paper for figures.