K. K. Golovkin

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Abstract

Full Text

K. K. Golovkin

Some Inequalities Between Norms of Mixed Derivatives of Functions of Several Variables

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Let us call a **functional of maximization type** (f.m.t.) over functions $\psi(h_1, \dots, h_n)$, defined for $h_i > 0$, any quantity admitting the representation

$$I[\psi(h_1, \dots, h_n)] = N[v(x_1, \dots, x_n)],$$

where $v(x_1, \dots, x_n) = \psi(e^{x_1}, \dots, e^{x_n})$, and N is some norm of functions in n-dimensional Euclidean space having the following properties:

1) $N[v(x+y)] \equiv [v(x)]$ (translation invariance);
2) if $|v_1(x)| \geq |v_2(x)|$, then $N[v_1] \geq N(v_2)$.

The most substantial theory uses a somewhat narrower class of f.m.t.' s, corresponding to norms N such that, along with translation invariance, they possess permutation invariance, i.e. $N[u(x)] = N[v(x)]$, if $u(x)$ and $v(x)$ are equimeasurable. Such I will be called **special** f.m.t.' s (or s.f.m.t.' s). An important example of an s.f.m.t. is

$$I_p[\psi] = \left(\int_0^\infty \cdots \int_0^\infty |\psi(h_1, \dots, h_n)|^p h_1^{-1} \cdots h_n^{-1}\, dh_1 \cdots dh_n\right)^{1/p}, \qquad (1)$$

$$1 \leq p \leq \infty$$

where in the case $p = \infty$, I_p is to be understood as $\sup_{h_i > 0}$.

The definition given is a direct generalization of the notion of a one-dimensional f.m.t. formulated by us earlier (see $(^{1-3})$).

Let $\|\cdot\|$ be any translation-invariant norm of functions of n variables. We shall say that $u(x) \in C(I, \bar{r})$ if, for sufficiently large m, the quantity

$$I\left[h_1^{-r_1} \cdots h_n^{-r_n} \left\|\binom{\bar{m}}{\bar{h}} u\right\|\right] < \infty, \qquad (2)$$

where

$$\binom{\bar{m}}{\bar{h}} = \Delta^{m_1}_{h_1,1}\Delta^{m_2}_{h_2,2}\cdots\Delta^{m_n}_{h_n,n},$$

and each of the operators Δ is the operation of taking a finite difference of the corresponding order m_i and step h_i with respect to the i-th variable. The set $Q_I(u)$ of all points in the space $\bar{r}$ with nonnegative coordinates for which the given function $u(x) \in C(I, \bar{r})$ will be called the **maximal I-body of smoothness** of $u(x)$, and its open part $\Omega_I(u)$ the **maximal open I-body of smoothness**. The intersection $\bigcap_I \Omega_I(u)$, taken over all s.f.m.t.' s, will be called the **maximal open body of smoothness** of $u(x)$ and denoted by $\Omega(u)$.

Theorem 1. For any f.m.t. I, $Q_I(u)$ is convex.

Theorem 2. If I is any *φ.t.m.*, then $Q_I(u) \subseteq Q_{I_\infty}(u)$. If I is any *s.φ.t.m.*, then $Q_{I_1}(u) \subseteq Q_I(u)$.

Denote by $D^{\bar{r}}u$ the mixed derivative of order r_1 with respect to x_1, r_2 with respect to x_2, etc.; here the r_i may be either integers or nonintegers. It is very important to be able to estimate norms of derivatives in terms of norms of the form (2). Such estimates, however, require in one form or another an "exclusion of polynomials," i.e., certain assumptions concerning the behavior of functions at infinity. We formulate these assumptions. Suppose first that $\bar{r}$ has integer components. Assume that for any $h_2, \dots, h_n$ the relation

$$\rho^{-r_1}\left\|\Delta^{r_1}_{\rho,1}\Delta^{r_2}_{h_2,2}\cdots\Delta^{r_n}_{h_n,n}u\right\| \underset{\rho\to\infty}{\longrightarrow} 0$$

holds.

Suppose, moreover, that for every $\varepsilon > 0$, for each $s \le n$, and for arbitrary $h_{s+1}, \dots, h_n$ the relation

$$\rho^{-r_1}\int_\varepsilon^\infty \cdots \int_\varepsilon^\infty \left\|\Delta^{r_1+1}_{\xi_1,1}\cdots\Delta^{r_{s-1}+1}_{\xi_{s-1},s-1}\Delta^{r_s}_{\rho,s}\Delta^{r_{s+1}}_{h_{s+1},s+1}\cdots\Delta^{r_n}_{h_n,n}u\right\|\times$$

$$\times\ \xi_1^{-1}\cdots\xi_{s-1}^{-1}\, d\xi_1\cdots d\xi_{s-1} \underset{\rho\to\infty}{\longrightarrow} 0.$$

We shall say that the function $u(x)$ satisfies condition (A) with respect to $\bar{r}$ if it satisfies this system of conditions, or any similar one obtained from it by a permutation of the indices.

Let a vector $\bar{r}$ with positive components be given. Represent it in the form $\bar{l}-\bar{\alpha}$, where $\bar{l}$ has integer components, and the components of $\bar{\alpha}$ lie in the interval $[0, 1)$. We shall say that $u(x)$ **satisfies condition (A) with respect to** $\bar{r}$ if $D^{-\bar{\alpha}}u$ satisfies condition (A) with respect to $\bar{l}$. Here $D^{-\bar{\alpha}}$ is the operator of mixed fractional integration.

A simple special case of satisfaction of condition (A) with respect to any $\bar{r}$ is the case when $\|u\| < \infty$.

In the class of functions satisfying condition (A) with respect to $\bar{r}$, the estimate

$$\left\|D^{\bar{r}}u\right\| \le CI_1\left[h_1^{-r_1}\cdots h_n^{-r_n}\left\|\left(\frac{\bar{m}}{\bar{h}}\right)u\right\|\right], \tag{3}$$

holds, where $m_i > r_i$, and the constant depends only on the indices r_i and m_i.

Theorem 3. In order that $\|D^{\bar{r}}u\| < \infty$, it is necessary that the point $(r_1, \dots, r_n)$ belong to $Q_{I_\infty}(u)$, and it is sufficient that it belong to $Q_{I_1}(u)$ and that $u(x)$ satisfy condition (A) with respect to $\bar{r}$. In this case estimate (3) is valid.

Theorem 4. Suppose the system of $n+1$ equations

$$\sum_{i=1}^{n+1}\mu_i = 1, \qquad \sum_{i=1}^{n+1}\mu_i\bar{r}^{(i)} = \bar{r}, \tag{4}$$

has a unique solution in positive μ_i. Then the estimate

$$I_1\left[h_1^{-r_1}\cdots h_n^{-r_n}\left\|\left(\frac{\bar{m}}{\bar{h}}\right)u\right\|\right] \le C\prod_{i=1}^{n+1} I_\infty^{\mu_i}\left[h_1^{-r_1^{(i)}}\cdots h_n^{-r_n^{(i)}}\left\|\left(\frac{\bar{m}}{\bar{h}}\right)u\right\|\right], \tag{5}$$

is valid, and, when condition (A) with respect to $\bar{r}$ is fulfilled, the estimate

$$\left\|D^{\bar{r}}u\right\| \le C\prod_{i=1}^{n+1}\left\|D^{\bar{r}^{(i)}}u\right\|^{\mu_i}. \tag{6}$$

The constants depend only on $\bar{r}, \bar{r}^{(i)}, \bar{m}$.

This theorem, together with Theorem 2, gives the following assertion.

Theorem 5. *Maximal open l-bodies of smoothness corresponding to different s.f.p.t.m. coincide with one another and, consequently, with $\Omega(u)$. The maximal open body of smoothness is convex.*

The question arises whether every convex domain is a maximal open body of smoothness of some function. The following theorem gives a partial answer to this question.

Theorem 6. *If $n = 2$, $\|\cdot\| = \|\cdot\|_{L_p}$, $1 \le p \le \infty$, then every convex polygon in the first quadrant of the plane (r_1, r_2) is a maximal open body of smoothness of some function $u(x_1, x_2)$.*

Theorem 7. *Suppose the system of $n+2$ equations*

$$\sum_{i=1}^{n+2}\mu_i = 1, \qquad \sum_{i=1}^{n+2}\mu_i p_i^{-1} = p^{-1} \quad (1 \le p, p_i \le \infty), \qquad \sum_{i=1}^{n+2}\mu_i \bar{r}^{(i)} = \bar{r} \tag{7}$$

has a unique solution in positive μ_i. *Then the estimate*

$$I_1\left[h_1^{-r_1}\cdots h_n^{-r_n}\left\|\left(\frac{\bar{m}}{\bar{h}}\right)u\right\|_p\right] \le C\prod_{i=1}^{n+2} I_\infty^{\mu_i}\left[h_1^{-r_1^{(i)}}\cdots h_n^{-r_n^{(i)}}\left\|\left(\frac{\bar{m}}{\bar{h}}\right)u\right\|_{p_i}\right], \tag{8}$$

is valid, and, when condition (A) is satisfied with respect to $\bar{r}$, *the estimate*

$$\|D^{\bar{r}}u\|_p \le C\prod_{i=1}^{n+2}\|D^{\bar{r}^{(i)}}u\|_{p_i}^{\mu_i}. \tag{9}$$

The constants depend only on $\bar{r}, \bar{r}^{(i)}, p, p_i, \bar{m}$.

It is of interest to ask to what extent the assertions of Theorems 6 and 7 remain valid if the algebraic systems (4) and (7) have a positive solution, but it is not unique, or if these systems have a unique but degenerate solution, i.e., some μ_i become zero. A number of examples based on the idea of logarithmic dimension (see ([4])) show that estimates (5) and (8) then, generally speaking, fail. However, estimates (6) and (9) remain valid in a number of cases. On the other hand, special cases of degeneration are known in which they too fail. Thus, the example of the function $xy\log(x^2+y^2)$ shows that boundedness of the second unmixed derivatives does not imply boundedness of the second mixed derivative. (Another function with the same property was constructed by B. S. Mityagin ([5]).) Consequently, estimate (6) is not always true in the degenerate case.

Leningrad Branch
of the V. A. Steklov Mathematical Institute
Academy of Sciences of the USSR

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Note: Figure translations are in progress. See original paper for figures.

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