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Abstract

Full Text

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THE RANDOM-PHASE APPROXIMATION IN THE THEORY OF A NONIDEAL BOSE GAS

(Presented by Academician N. N. Bogolyubov, 30 VI 1964)

The basic principles of the theory of nonideal Bose systems were developed in 1947 in the work of N. N. Bogolyubov ([1]), in which a system of weakly interacting bosons at zero temperature was considered. The present and subsequent papers are devoted to the study of a Bose system at temperatures different from zero, within the framework of the approximation used in ([1]).

Let us consider a system of N Bose particles contained in a volume V, interacting pairwise with one another through a potential $U(x)$. The Green function $\langle\langle a_p^+ a_{p+q};\ a_{p'+q}^+ a_{p'}\rangle\rangle$, where a_p and a_p^+ are Bose annihilation and creation operators for particles in a state with momentum p, satisfies the equation

$$\begin{aligned} i\frac{d}{dt}\langle\langle a_p^+ a_{p+q};\ a_{p'+q}^+ a_{p'}\rangle\rangle &= \delta(t-t')\delta_{pp'}(n_p - n_{p+q}) \\ &+ \left(\frac{q^2}{2m} + \frac{p\cdot q}{m}\right)\langle\langle a_p^+ a_{p+q};\ a_{p'+q}^+ a_{p'}\rangle\rangle \\ &+ \frac{1}{V}\sum_{k\neq 0}\nu(k)\langle\langle a_p^+ \rho_k a_{p+q-k} - a_{p+k}^+ \rho_k a_{p+q};\ a_{p'+q}^+ a_{p'}\rangle\rangle. \end{aligned} \tag{1}$$

Here, in the case, for example, of retarded functions, $\langle\langle A;\ B\rangle\rangle = -i\theta(t-t')\langle[A(t);\ B(t')]\rangle$; $A(t)$ and $B(t')$ are operators in the Heisenberg representation; $n_p = \langle a_p^+ a_p\rangle$; $\nu(k) = \int e^{-ikx}U(x)\,dx$; $\rho_k = \sum_{p_1} a_{p_1}^+ a_{p_1+k}$ is the density operator. In the right-hand side of equation (1) the term with $k = q$ can be singled out from the sum.

Discarding the remaining terms of the sum (with $k \neq 0, q$), we obtain the model equation of the random-phase approximation

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$$
\begin{aligned}
i\frac{d}{dt}\langle\langle a_p^+ a_{p+q};\ a_{p'+q}^+ a_{p'}\rangle\rangle = \delta(t-t')\delta_{pp'}(n_p - n_{p+q}) \\
+ \left(\frac{q^2}{2m} + \frac{p\cdot q}{m}\right)\langle\langle a_p^+ a_{p+q};\ a_{p'+q}^+ a_{p'}\rangle\rangle \\
+ \frac{n_p - n_{p+q}}{V}\,\nu(q)\,\langle\langle \rho_q;\ a_{p'+q}^+ a_{p'}\rangle\rangle,
\end{aligned}
\tag{2}
$$

where we have additionally carried out a linearization by pairing the operators a_p^+, a_p and a_{p+q}^+, a_{p+q} and replacing them by their mean occupation numbers $n_p = \langle a_p^+ a_p\rangle$. The random-phase approximation corresponds to the approximation of weak interaction and high density. In the case of strong interaction, one must also take into account the discarded terms with $k \neq 0, q$. Assuming the gas density to be small, one may introduce the scattering matrix in the usual way (see, for example, ([2])); as a result one obtains an equation of the same form as (2), in which the Fourier component of the interaction potential $\nu(q)$ must be replaced by the scattering amplitude f ($\nu \to f/m$). Since in what follows we shall not go beyond the framework of equation (2), the results given below, with the indicated replacement, also carry over to the case of a low-density gas with an interaction of the hard-sphere type.

Put in equation (2) $p = 0,\ p' = 0$ and $p = -q,\ p' = 0$. Taking into account the assumption of N. N. Bogolyubov ([1]) that below the Bose-condensation temperature, in the case of weak interaction, $a_0 = \sqrt{N_0},\ a_0^+ = \sqrt{N_0}$, where N_0 is the number of particles in the state with momentum $p = 0$, of the order of the total

of particles N ($N_0 \sim N \gg n_p,\ p \neq 0$), we obtain, for the Fourier components of the one-particle Green' s functions $\langle\langle a_q; a_q^+\rangle\rangle$ and $\langle\langle a_{-q}^+; a_q^+\rangle\rangle$, the equations

$$
\begin{aligned}
\left(E - \frac{q^2}{2m}\right)\langle\langle a_q \mid a_q^+\rangle\rangle = 1 + \frac{\sqrt{N_0}}{V}\,v(q)\langle\langle \rho_q \mid a_q^+\rangle\rangle, \\
\left(E + \frac{q^2}{2m}\right)\langle\langle a_{-q}^+ \mid a_q^+\rangle\rangle = -\frac{\sqrt{N_0}}{V}\,v(q)\langle\langle \rho_q \mid a_q^+\rangle\rangle,
\end{aligned}
\tag{3}
$$

where

$$
\langle\langle A; B\rangle\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iE(t-t')}\langle\langle A \mid B\rangle\rangle_E\, dE; \qquad \langle\langle A \mid B\rangle\rangle_E = \int_{-\infty}^{\infty} e^{iEt}\langle\langle A(t); B(0)\rangle\rangle\, dt
$$

and, in the case of the retarded function, $\operatorname{Im} E > 0$. The density operator is equal to

$$
\rho_q = \sqrt{N_0}\,(a_q + a_{-q}^+) + \sum_{p_1 \neq 0,\,-q} a_{p_1}^+ a_{p_1+q}.
$$

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At zero temperature, because of the strong condensation, the states with $p \neq 0$ are unimportant, and one may put $\rho_q \simeq \sqrt{N_0}(a_q + a^+_{-q})$. Substituting this value of the operator ρ_q into (3), we obtain equations corresponding to N. N. Bogolyubov' s theory ([1]), from which it is easy to obtain the spectrum of elementary excitations

$$E_q = \sqrt{\left(\frac{q^2}{2m}\right)^2 + \frac{q^2}{m}\frac{N_0}{V}v(q)},$$

where $N_0 \simeq N$. At temperatures different from zero, both parts of the operator ρ_q will give contributions of the same order. Therefore the Green' s function $\langle\langle \rho_q; a^+_q \rangle\rangle$ must be found from its own equation. Performing in (2) a Fourier transformation with respect to time, expressing the function $\langle\langle a^+_p a_{p+q}; a^+_q a_0 \rangle\rangle$ in terms of $\langle\langle \rho_q; a^+_q a_0 \rangle\rangle$, and then summing over p, we obtain the following expression for the function $\langle\langle \rho_q \mid a^+_q \rangle\rangle_E$:

$$\langle\langle \rho_q \mid a^+_q \rangle\rangle_E = \frac{\sqrt{N_0}}{E - \frac{q^2}{2m}} \left[1 - \frac{v(q)}{V}\sum_p \frac{n_p - n_{p+q}}{E - \frac{q^2}{2m} - \frac{p\cdot q}{m}}\right]^{-1} =$$

$$= \sqrt{N_0}\left(E + \frac{q^2}{2m}\right)\left[E^2 - \left(\frac{q^2}{2m}\right)^2 - \frac{q^2}{m}\frac{N}{V}v(q) - 2\frac{v(q)}{V}\sum_{p\neq 0} n_p \frac{p\cdot q}{m}\frac{E^2 + \frac{q^2}{2m}\left(\frac{q^2}{2m} + \frac{p\cdot q}{m}\right)}{E^2 - \left(\frac{q^2}{2m} + \frac{p\cdot q}{m}\right)^2}\right]^{-1}. \tag{4}$$

In expression (4) the last term in square brackets is proportional to the mean kinetic energy

$$\overline{p^2}/2m = N^{-1}\sum_{p\neq 0}(p^2/2m)n_p,$$

per particle. At zero temperature this quantity is small, since in the overcondensate state (with $p \neq 0$) there is a small fraction of all particles. We shall assume that the mean kinetic energy of a particle is also small in comparison with the potential energy near the Bose-condensation temperature θ_{cr}. In this case

$$\overline{p^2}/2m \sim \theta_{\text{cr}} \sim 1/2ma^2,$$

where $a^3 = V/N$, and the last condition is written in the form $v(0)/V \gg \theta_{\text{cr}}$, or $v(0)m/a \gg 1$. If the density is sufficiently large ($a/d \ll 1$, d being the

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radius of action of the forces), this condition will not contradict the condition for applicability of the Born approximation $v(0)m/d \ll 1$.

Now discarding in (4), as small, the last term in square brackets, from (3) we obtain the following expressions for the one-particle Green' s functions:

$$\langle\langle a_q \mid a_q^+ \rangle\rangle = \frac{1}{E - \dfrac{q^2}{2m}} \left\{ 1 + \frac{N_0}{V} \frac{v(q)\left(E + \dfrac{q^2}{2m}\right)}{E^2 - E_q^2} \right\} = \frac{1 - \dfrac{N_0}{N}}{E - q^2/2m} + \frac{N_0}{N} \frac{E + \dfrac{q^2}{2m} + \dfrac{N}{V} v(q)}{E^2 - E_q^2}, \tag{5}$$

$$\langle\langle a_{-q}^+ \mid a_q^+ \rangle\rangle = -\frac{N_0}{V} \frac{v(q)}{E^2 - E_q^2}, \qquad \text{where } E_q = \sqrt{\left(\frac{q^2}{2m}\right)^2 + \frac{q^2}{m}\frac{N}{V} v(q)}.$$

For the function $\langle\langle \rho_q \mid \rho_{-q} \rangle\rangle_E$, in the same approximation, we have the already known expression ([3])

$$\langle\langle \rho_q \mid \rho_{-q} \rangle\rangle = \frac{\dfrac{q^2}{m} N}{E^2 - E_q^2}. \tag{6}$$

Expressions (5) and (6), with the reservations made, are valid over the entire temperature interval θ from zero to θ_{cr}. At $\theta = 0$ formulas (5) coincide with the corresponding results of the theory based on the model Hamiltonian of N. N. Bogolyubov ([1]), since the number of particles in the condensate N_0 practically coincides with the total number of particles N. At $\theta = \theta_{\text{cr}}$, when $N_0 = 0$, from (5) we obtain expressions for the Green functions of the ideal gas. With the aid of the usual spectral formulas relating Green functions to correlation functions (see, for example, ([4])), from (5) and (6) we obtain

$$n_q = \langle a_q^+ a_q \rangle = \left(1 - \frac{N_0}{N}\right)\left(e^{q^2/2m\theta} - 1\right)^{-1} + \frac{N_0}{2N}\left(\frac{\dfrac{q^2}{2m} + \dfrac{N}{V} v(q)}{E_q} \operatorname{cth} \frac{E_q}{2\theta} - 1\right),$$

$$\langle a_q^+ a_{-q}^+ \rangle = -\frac{N_0}{V} v(q) \operatorname{cth} \frac{E_q}{2\theta}, \qquad \langle \rho_{-q} \rho_q \rangle = \frac{q^2}{m} \frac{N}{2E_q} \operatorname{cth} \frac{E_q}{2\theta}. \tag{7}$$

From the equation $N = N_0 + \sum_{q \neq 0} n_q$ we find a satisfactory dependence of the number of particles in the condensate N_0 on temperature,

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$$\frac{N_0}{N} = \left(1 - \frac{\theta}{\theta_{\text{cr}}}\right)^{3/2} \left[1 + \frac{1}{2N} \sum_{q \neq 0} \left(\frac{\frac{q^2}{2m} + \frac{N}{V} v(q)}{E_q} \operatorname{cth} \frac{E_q}{2\theta} - \operatorname{cth} \frac{q^2}{4m\theta} \right)\right]^{-1}, \tag{8}$$

where the critical temperature is

$$\theta_{\text{cr}} = \frac{1}{2ma^2} \left(\frac{1}{2\pi^2} \int_0^\infty \frac{x^2\, dx}{e^{x^2} - 1} \right)^{-2/3}$$

and coincides with the Bose-condensation temperature in an ideal gas.

The expressions obtained for the Green functions have, however, one substantial shortcoming. The one-particle Green function $\langle\langle a_q \mid a_q^+ \rangle\rangle$ has a pole $E = q^2/2m$, which does not satisfy L. D. Landau' s superfluidity criterion ([5]). The "exact" equations (3) and (4), which hold before neglecting terms proportional to the mean kinetic energy per particle, do not lead to the appearance of this pole. Indeed, substituting $E = q^2/2m$ in (4), we obtain

$$\langle\langle \rho_q \mid a_q^+ \rangle\rangle_E \Big|_{E=q^2/2m} = \frac{\sqrt{N_0 q^2/m}}{-\frac{q^2}{m}\frac{N}{V} v(q) + \frac{q^2}{m} v(q) \frac{1}{V} \sum_{p \neq 0} n_p} = -\frac{V}{\sqrt{N_0}} \frac{1}{v(q)}, \tag{9}$$

as a result of which the right- and left-hand sides of the first of equations (3) simultaneously vanish. A successive approximate allowance for the discarded terms leads to the appearance in the Green functions, instead of the pole $E = q^2/2m$, of poles of the second-sound type with a propagation velocity proportional to the mean kinetic energy of one particle. This result cannot be obtained by directly expanding expression (4) in powers of $\overline{p^2}/2m$ with subsequent discarding of higher terms, since the expansion parameter

$$\frac{q^2}{m} \frac{\overline{p^2}}{2m} \Big/ \left[\left(1 - \frac{N_0}{N}\right) \left(E^2 - \left(\frac{q^2}{2m}\right)^2 \right) \right]$$

will be of order unity for E lying in the vicinity of the pole.

Let us note that to the Green functions (5) and (6) there correspond the following approximate expressions for the operators $a_q(t)$ and $\rho_q(t)$ in the Heisenberg repre-

representation:

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$$
\begin{aligned}
a_q(t) &\simeq \sqrt{1-\frac{N_0}{N}}\,\alpha_q e^{-i\frac{q^2}{2m}t} + \sqrt{\frac{N_0}{N}}\left(u_q\beta_q e^{-iE_{qt}} + v_q\beta^+_{-q}e^{iE_{qt}}\right),\\
\rho_q(t) &\simeq \sqrt{N}\,(u_q+v_q)\left(\beta_q e^{-iE_{qt}} + \beta^+_{-q}e^{iE_{qt}}\right),\\
\rho'_q(t) \equiv i\frac{d\rho_q}{dt} &\simeq \sqrt{N}\,(u_q+v_q)E_q\left(\beta_q e^{-iE_{qt}} - \beta^+_{-q}e^{iE_{qt}}\right).
\end{aligned}
\tag{10}
$$

where α_q and β_q, in the approximation under consideration, are Bose operators of two quasiparticle systems ($[\alpha,\beta^+]=[\alpha^+,\beta^+]=0$),

$$
E_q = \sqrt{\left(\frac{q^2}{2m}\right)^2 + \frac{q^2}{m}\frac{N}{V}v(q)},
$$

$$
u_q^2 = 1 + v_q^2 = \frac{1}{2}\left(1 + \frac{\frac{q^2}{2m} + \frac{N}{V}v(q)}{E_q}\right)
$$

are the parameters of the canonical transformation of N. N. Bogoliubov [(1)], in which N_0 has been replaced by N. Computing directly with the aid of (10) $\langle\langle a_q; a_q^+\rangle\rangle = -i\theta(t-t')\langle[a_q(t); a_q^+(t')]\rangle$, etc., and passing to the Fourier components $\langle\langle a_q|a_q^+\rangle\rangle_E, ...$, we obtain (5) and (6). Setting $t=0$ in (10) and regarding the resulting relations as exact (in this case the operators α_q and β_q will no longer be exactly Bose operators), we can find expressions for α_q and β_q in terms of a_q, ρ_q, and $\rho'_q = \sum_p\left(\frac{q^2}{2m} + \frac{p\cdot q}{m}\right)a_p^+ a_{p+q}$:

$$
\alpha_q = \frac{1}{\sqrt{1-N_0/N}}\left\{a_q - \frac{\sqrt{N_0}}{2N}\left(\rho_q + \frac{2m}{q^2}\rho'_q\right)\right\},
$$

$$
\beta_q = \frac{1}{2\sqrt{N}}\,\frac{1}{u_q+v_q}\left(\rho_q + \frac{1}{E_q}\rho'_q\right). \tag{11}
$$

It is easy to verify that the commutation relations for the operators α_q and β_q will be Bose ones only on the average over the statistical ensemble:

$$
\langle[\alpha_q;\alpha_{q'}^+]\rangle = \delta_{qq'}, \quad \langle[\beta_q;\beta_{q'}^+]\rangle = \delta_{qq'}, \quad \langle[\alpha_q;\alpha_{q'}]\rangle = \langle[\beta_q;\beta_{q'}]\rangle = \langle[\alpha_q;\beta_{q'}]\rangle = \langle[\alpha_q;\beta_{q'}^+]\rangle = 0.
$$

The operators α_q and β_q defined by expressions (11) prove useful in calculating Green' s functions in the following approximation.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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