Reports of the Academy of Sciences of the USSR

MATHEMATICS

1964

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR
1964. Volume 154, No. 5

MATHEMATICS

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ON A METHOD FOR ESTIMATING RESOLVENTS OF NON-SELF-ADJOINT OPERATORS

(Presented by Academician L. S. Pontryagin, 24 X 1963)

In this note we shall try, by means of examples, to set forth the idea of a method that makes it possible to estimate the resolvents of operators acting in abstract spaces. By this method, in particular, Lemma 2 of ([1]) was proved. Following ([2]), by $\mathfrak{S}_\infty$ we shall denote the Banach space of all completely continuous operators acting in a separable Hilbert space $\mathfrak{H}$. With an operator $A \in \mathfrak{S}_\infty$ we shall associate the sequences $\mu_n(A)$ and $s_n(A)$ $(n = 1, 2, ...)$ of eigenvalues of the operators A and $(A^*A)^{1/2}$, numbered in decreasing order of their moduli with multiplicities taken into account, and the functions $n(t, A)$ and $\nu(t, A)$ $(t > 0)$, giving the number of terms of these sequences greater than $1/t$. By $C(\alpha, \beta)$ we shall denote any positive constant depending only on the parameters standing in parentheses.

We begin by establishing an auxiliary estimate for the resolvent of an n-dimensional operator T. Put, for an integer $p \geq 1$,

$$D_0(T, p) = \det(I - T) \exp\left(\operatorname{sp} \sum_{l=1}^{p-1} l^{-1} T^l\right),$$

$$D_1(T, p) = D_0(T, p)((I - T)^{-1} f, g),$$

where by $\det(I - T)$ is meant the determinant of the restriction of the operator $I - T$ to the range of the operator T, and f and g are vectors of the space $\mathfrak{H}$, $\|f\| = \|g\| = 1$.

Lemma 1. The estimates

$$\ln |D_k(T, p)| \leq C(p) \left(k + \sum_{j=1}^{n} G(s_j(T), p)\right) \qquad (k = 0, 1),$$

hold, where $G(u,1)=\ln(1+u)$, $G(u,p)=(1+u)^{-1}u^p$ for $p\geq 2$.

Proof. We consider the more difficult case $k=1$. Extend the space $\mathfrak{H}$ by adjoining to it a unit vector e, orthogonal to $\mathfrak{H}$, to a Hilbert space $\mathfrak{H}_1$ with scalar product $\langle\cdot,\cdot\rangle$, and introduce the operator T_1

$$T_1=\widetilde{T}-\langle\cdot,e\rangle f-\langle\cdot,g\rangle e+\langle\cdot,e\rangle e,\qquad \widetilde{T}\supset T,\quad \widetilde{T}e=0.$$

Let

$$D_2(T)=\det(E-T_1)\exp\left(\operatorname{sp}\sum_{l=1}^{p-1}l^{-1}T_1^l\right),$$

where E is the identity operator in $\mathfrak{H}_1$. In view of the equality

$$\det(E-T_1)=-\det(I-T)((I-T)^{-1}f,g),$$

we have

$$\ln|D_1(T,p)|\leq\ln|D_2(T)|+C(p)(1+s_1^{p-1}(T)).\tag{1}$$

Next, using the known estimate of the primary factor of the canonical product (5) and Weyl' s inequality (6), we have

$$\begin{aligned}\ln|D_2(T)|&=\sum_{\mu_j(T_1)}\left(\ln|1+\mu_j(T_1)|+\operatorname{Re}\sum_{l=1}^{p-1}l^{-1}\mu_j^l(T_1)\right)\\&\leq C(p)\sum_{\mu_j(T_1)}G(|\mu_j(T_1)|,p)\leq C(p)\sum_{s_j(T_1)}G(s_j(T_1),p)\\&\leq C(p)\left(1+\sum_{j=1}^{n}G(s_j(T),p)\right).\end{aligned}\tag{2}$$

Combining (1) and (2), we obtain the assertion of Lemma 1 for $k=1$. We note that I. Ts. Gohberg and M. G. Krein, by means of an elegant device, found the exact form of Lemma 1 for $p=1$ (see also (4)).

The simplest result obtained by our method is an estimate of the resolvent of a completely continuous operator in terms of its s-numbers. Estimates of this kind were obtained earlier ([3],[4],[7]) for operators of finite order (i.e., operators for which the function $\nu(t)$ has a power majorant). Our estimate, while refining these results, is at the same time also suitable for an operator of infinite order.

In order to measure the growth of the resolvent, we introduce the function $T(r, A)$, which is an analogue of the Nevanlinna characteristic of growth of scalar meromorphic functions:

$$T(r, A) = \frac{1}{2\pi} \int_0^{2\pi} \ln^+ \|(I - re^{i\theta} A)^{-1}\| \, d\theta + \int_0^r \frac{n(t, A)}{t} \, dt.$$

Theorem 1. *For any operator $A \in \mathfrak{S}_\infty$ the estimate*

$$T(r, A) \leq C(\varepsilon) \left(1 + \int_0^{r(1+\varepsilon)} \frac{\nu(t, A)}{t} \, dt \right)$$

holds.

Proof. Take the number $R = r(1 + \delta)$, $\delta > 0$, and put $\nu(R) = m$. Using the decomposition $A = UH$, where U is an isometric operator and $H = (A^* A)^{1/2}$, one can represent the operator in the form $A = A_m + E_m$, where A_m is an m-dimensional operator whose s-numbers coincide with the s-numbers of the operator A that are greater than $1/R$, and $\|E_m\| \leq 1/R$.

We transform the resolvent of the operator A to the form

$$(I - \lambda A)^{-1} = R_\lambda^{(m)} (I - \lambda A_{mR_\lambda}^{(m)})^{-1}, \qquad \text{where } R_\lambda^{(m)} = (I - \lambda E_m)^{-1}.$$

Since $\|R_\lambda^{(m)}\| \leq (1 + \delta)/\delta$ for $|\lambda| \leq r$, then, introducing the notation $D_k(\lambda) = D_k(\lambda A_{mR_\lambda}^{(m)}, 1)$ $(k = 0, 1)$, we have, according to Lemma 1, for $p = 1$,

$$\ln |(D_k(\lambda)| \leq C \left(k + \int_0^R \ln \left(1 + \frac{1 + \delta}{\delta t} \right) d\nu(t, A) \right) \leq$$

$$\leq C(\delta) \left(k + \int_0^{R(1+\delta)} \frac{\nu(t, A)}{t} \, dt \right) \qquad (k = 0, 1).$$

We now apply Jensen' s formula to the function $D_0(\lambda)$:

$$\int_0^r \frac{n(t, A)}{t} \, dt + \frac{1}{2\pi} \int_0^{2\pi} \ln^- |D_1(r, e^{i\theta})| \, d\theta =$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \ln^+ |D_1(re^{i\theta})| \, d\theta \leq C(\delta) \int_0^{R(1+\delta)} \frac{\nu(t)}{t} \, dt.$$

Combining the estimates obtained and setting $(1 + \delta)^2 = 1 + \varepsilon$, we get

$$T(r,A) \leqslant \ln\frac{1+\delta}{\delta} + \int_0^r \frac{n(t,A)}{t}\,dt + \sup_{\|f\|=\|g\|=1} \frac{1}{2\pi}\int_0^{2\pi} \ln^+\left|\frac{D_2(re^{i\theta})}{D_1(re^{i\theta})}\right| d\theta \leqslant$$

$$\leqslant C(\varepsilon)\left(1 + \int_0^{r(1+\varepsilon)} \frac{v(t,A)}{t}\,dt\right).$$

Theorem 1 is proved completely.

Corollary 1. If $A \in \mathfrak{S}_p$ $(p > 0)$, i.e.

$$\sum_{k=1}^{\infty} s_k^p(A) < \infty,$$

then the characteristic of its resolvent has a monotone majorant $V(r)$ of convergent type of order p, i.e.

$$T(r) \leqslant V(r), \qquad \int^{\infty} \frac{V(r)}{r^{1+p}}\,dr < \infty.$$

Corollary 2. If $s_n(A) = O(n^{-1/p}L(n))$ $(p > 0)$, where $L(r)$ is a slowly varying function (this means that $rL'(r)/L(r) \to 0$ as $r \to \infty$), then

$$T(r,A) = O(r_1^{pL}(r)),$$

where $L_1(r)$ is a slowly varying function determined by the condition that the functions $r^{1/p}L(r)$ and $r_1^{pL}(r)$ are mutually inverse.

This proposition remains valid if everywhere in it the sign O is replaced by o.

Corollary 3. If $A \in \mathfrak{S}_\omega$, i.e.

$$\sum_{k=1}^{\infty}(2k-1)^{-1}s_k(A) < \infty,$$

then $T(r,A)$ has a monotone majorant $V(r)$ such that

$$\int^{\infty} \frac{\ln V(r)}{r^2}\,dr < \infty.$$

Corollary 4. If $A \in \mathfrak{S}_\infty$ and the vectors $f, g \in \mathfrak{H}$, $\|f\| = \|g\| = 1$, are such that $((I-\lambda A)^{-1}f, g)$ is an entire function of λ (this is satisfied, for example, if the operator A has a single point of spectrum), then the estimate

$$M(r) = \max\left|((I-\lambda A)^{-1}f, g)\right| \leqslant \exp\left(C(\varepsilon)T\left(r\left(1+\frac{\varepsilon}{2}\right)\right)\right) \leqslant$$

$$\leqslant \exp\left\{C(\varepsilon)\left(1+\int_0^{r(1+\varepsilon)}\frac{v(t,A)}{t}\,dt\right)\right\},$$

is valid, and, consequently, the assertions of Corollaries 1-3 remain in force upon replacing $T(r)$ in them by $\ln M(r)$.

It is curious that a weakened version of Theorem 1 is also true for Banach spaces, if for $s_n(A)$ one takes the error of the best approximation of the operator A by $(n-1)$-dimensional operators (as I. Ts. Gohberg and M. G. Krein showed, in the case of Hilbert space this definition coincides with the usual one).

Theorem 2. Let A be an operator acting in a Banach space. If $s_n(A)\to 0$, then the estimate

$$T(r,A)\leqslant C(\varepsilon)\left(1+v(r(1+\varepsilon))\ln\frac{r(1+\varepsilon)}{s_1(A)}\right).$$

is valid.

This theorem is proved analogously to Theorem 1, except that instead of H. Weyl' s inequalities, an analogue of which is unknown to the author, the trivial estimate $|\mu_k(T)|\leqslant s_1(T)$ is used.

If in the proof of Theorems 1 and 2 the function-theoretic apparatus is limited to the application of Jensen' s theorem, then in estimating the resolvents

operators in a domain with a discontinuous boundary, specific difficulties arise that are connected with estimating the quotient of two holomorphic functions having a prescribed majorant furnished by Lemma 1.

A typical proposition which has to be used in these cases is Lemma 2.

Lemma 2. *Let the functions* $\varphi_1(z)$, $\varphi_2(z)$ *and* $f(z)$ *be holomorphic in the half-plane* $\operatorname{Im} z>0$, *and let* $f(z)\varphi_2(z)=\varphi_1(z)$. *Suppose the inequalities* $\ln|\varphi_k(z)|\leqslant M(y^{-1})$ $(k=1,2)$ *hold, where* $M(t)$ *is a monotonically increasing function. If* $\varphi_2(z)\to 1$ *as* $z\to\infty$ *in any angle lying inside the upper half-plane, then for* $\alpha>1$ *the estimate*

$$\ln|f(z)|\leqslant\frac{C(\alpha)}{y^{\alpha}}\int_0^{C(\alpha)y^{-1}}\frac{M(t)}{t^{1+\alpha}}\,dt.$$

The proof of this lemma is close to the proof of Theorem 1 in (8). Using Lemma 2 and Lemma 1 (the latter this time in full generality), one can prove the following theorem.

Theorem 3. *Let* $F\in\mathfrak{S}_\infty$, *and let the resolvent of the operator* T *in the angle* $|\arg\lambda|<\gamma$ $(\gamma>0)$ *satisfy the estimate*

$$\|(I-\lambda T)^{-1}\|\leqslant a\sec\frac{\pi}{2\gamma}\varphi,\qquad \varphi=\arg\lambda.$$

Let vectors f and g ($\|f\| = \|g\| = 1$) be such that the function $(R_\lambda f, g) = ((I - \lambda A)^{-1} f, g)$, $A = T + F$, is holomorphic in the angle $|\varphi| < \gamma$. Then in this angle the estimate

$$\ln |(R_\lambda f, g)| \leqslant C\eta^{\frac{\pi}{2\gamma}(1+\delta)} \int_0^{C\eta} \frac{\nu(t, F)}{t^{1+\frac{\pi}{2\gamma}(1+\delta)}}\, dt,$$

holds, where $\eta = r \sec^{1+\delta} \frac{\pi}{2\gamma}\varphi$, $\delta > 0$, $C = C(\delta, \gamma, a)$.

From Theorem 3 it is easy to derive

Theorem 4. *Let $F \in \mathfrak{S}_\infty$, and let the resolvent of the operator T (in general, unbounded) satisfy, for $\operatorname{Im}\lambda > 0$, the estimate*

$$\|(A - \lambda I)^{-1}\| \leqslant \frac{a}{\operatorname{Im}\lambda}.$$

Let vectors f and g ($\|f\| = \|g\| = 1$) be such that the function $(R_\lambda f, g) = ((A - \lambda I)^{-1} f, g)$, $A = T + F$, is holomorphic for $\operatorname{Im}\lambda > 0$. Then in the upper half-plane, for $\varepsilon > 0$, the estimate

$$\ln |(R_\lambda f, g)| \leqslant C(\varepsilon, a) \left(1 + \frac{1}{y^{1+\varepsilon}} \int_0^{C(\varepsilon,a)y^{-1}} \frac{\nu(t, F)}{t^{2+\varepsilon}}\, dt\right)$$

holds.

Lemma 2 of (1) is a simple consequence of Theorem 4, but it is much less precise and is not required for the proof of Lemma 2 of the present paper.

The theorems proved do not exhaust the possibilities of our method. Resolvent estimates obtained with its aid find application in the proof of completeness theorems.

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Received
17 X 1963

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Note: Figure translations are in progress. See original paper for figures.