# Mathematics

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**Abstract**

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## Mathematics

**B. S. VERKHOVSKII**

# DISTRIBUTION OF HETEROGENEOUS PRODUCTS TAKING INTO ACCOUNT THE "PROCESSING" CAPACITY OF INTERMEDIATE POINTS

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An economic model is considered for the production of products of several types at a number of enterprises. In the process of delivery to consumers, the produced products undergo additional "processing" at intermediate points. The nature of this "processing" varies: 1) transshipment from one mode of transport to another; 2) storage; 3) assembly into sets; 4) sorting; 5) packaging; 6) a production process; 7) accumulation, etc.

Let $a_{ij}$ be the capacity of the $i$-th enterprise for producing the $j$-th type of product; $i = 1, \dots, m$; $j = 1, \dots, n$; $b_k$ the "processing" capacity of the $k$-th intermediate point; $c_{jl}$ the demand for the $j$-th type of product at the $l$-th consumption point; $k = 1, \dots, r$; $l = 1, \dots, s$.

Let $b_k$ be expressed in units of the reduced product. This is possible when, at one and the same intermediate point, the nature of the processing is the same for all types of products (at different intermediate points the nature of the processing may be different even for one and the same type of product).

Denote by $x_{ijkl}$ the quantity of the $j$-th product supplied along the route $i{-}k{-}l$; by $p_{ijkl}$, the costs of production, processing, and transportation of one unit of the $j$-th product along the route $i - k - l$. It is necessary to find $x_{ijkl} \geq 0$ satisfying the conditions

$$\sum_{k,l} x_{ijkl} \leq a_{ij}; \qquad \sum_{i,k} x_{ijkl} = c_{jl}; \tag{1}$$

$$\sum_{i,j,l} x_{ijkl} \leq b_k \tag{2}$$

and delivering a minimum to the functional

$$L = \sum_{i,j,k,l} x_{ijkl} p_{ijkl}. \tag{3}$$

In order that problem (1)–(3) have a solution, it is necessary and sufficient that the conditions

$$\sum_{l=1}^{s} c_{jl} \le \sum_{i=1}^{m} a_{ij}$$

hold for all $j = 1, \dots, n$;

$$\sum_{j=1}^{n} \sum_{l=1}^{s} c_{jl} \le \sum_{k=1}^{r} b_k.$$

Necessity follows from the fact that the results of summation do not depend on the order of summation. Sufficiency follows from the fact that

$$x_{ijkl} = a_{ij} b_k c_{jl} \Big/ \sum_{i=1}^{m} a_{ij} \sum_{k=1}^{n} b_k$$

satisfies all conditions (1)–(2).

In what follows we assume that in (1) and (2) the equality sign holds. We shall show that a basic solution of the problem with conditions (1), (2) has no more than $(m + s - 1)n + r - 1$ nonzero components.

Since $\sum_{l,i,k} x_{ijkl} = \sum_{i,k,l} x_{ijkl}$, we have

$$\sum_{k,l} x_{1jkl} = \sum_{l,i,k} x_{ijkl} - \sum_{i=2}^{m} \left( \sum_{k,l} x_{ijkl} \right).$$

Similarly, since $\sum_{j,l,i,k} x_{ijkl} = \sum_{k,i,j,l} x_{ijkl}$, we have

$$\sum_{i,j,l} x_{ij1l} = \sum_{j,l,i,k} x_{ijkl} - \sum_{k=2}^{r} \left( \sum_{i,j,l} x_{ijkl} \right).$$

Thus, the $(n + 1)$ conditions $\sum_{k,l} x_{1jkl} = a_{1j}$; $j = 1, \dots, n$; $\sum_{i,j,l} x_{ij1l} = b_1$ are dependent, and since the total number of constraints is $(m + s)n + r$, the number of independent constraints is no more than $(m + s)n + r - n - 1$. The rest follows from the general theory of linear programming.

In order to ensure nondegeneracy of the problem, let us impose small perturbations on the components of the constraint vector. To this end, consider $\tilde{a}_{ij}, \tilde{b}_k, \tilde{c}_{jl}$ instead of the corresponding $a_{ij}, b_k, c_{jl}$:

$$\tilde{a}_{ij} = a_{ij} + \varepsilon^{i+mrs(j-1)} \frac{1-\varepsilon^{mrs}}{1-\varepsilon^{m}},$$

$$\tilde{b}_k = b_k + \varepsilon^{m(k-1)+1} \frac{1-\varepsilon^{m}}{1-\varepsilon} \frac{1-\varepsilon^{mnrs}}{1-\varepsilon^{mr}},$$

$$\tilde{c}_{jl} = c_{jl} + \varepsilon^{mr(l-1)+mrs(j-1)+1} \frac{1-\varepsilon^{mr}}{1-\varepsilon}.$$

Algorithm for finding a basic solution:

$$x_{1111} = \min(a_{11}, b_1, c_{11}).$$

Denote $\hat{a}_{11} = a_{11} - x_{1111}$, $\hat{b}_1 = b_1 - x_{1111}$, $\hat{c}_{11} = c_{11} - x_{1111}$. At least one of the three quantities $\hat{a}_{11}, \hat{b}_1, \hat{c}_{11}$ is equal to zero. Suppose $x_{i_1 j_1 k_1 l_1} = \min(\hat{a}_{i_1 j_1}, \hat{b}_{k_1}, \hat{c}_{j_1 l_1})$ has been found, where $\hat{a}_{i_1 j_1}, \hat{b}_{k_1}, \hat{c}_{j_1 l_1}$ are the current values. We proceed to finding $x_{i_2 j_2 k_2 l_2}$, where $i_2, j_2, k_2, l_2$ are determined from the relations:

$$i_2 + m(j_2 - 1) = i_1 + m(j_1 - 1) + E\left(\frac{1}{1 + \hat{a}_{i_1 j_1} - x_{i_1 j_1 k_1 l_1}}\right),$$

$$k_2 = \begin{cases} k_1, & \text{if } \hat{b}_{k_1} > x_{i_1 j_1 k_1 l_1}, \\ k_1 + 1, & \text{if } \hat{b}_{k_1} = x_{i_1 j_1 k_1 l_1}, \end{cases}$$

$$l_2 + s(j_2 - 1) = l_1 + s(j_1 - 1) + E\left(\frac{1}{1 + \hat{c}_{j_1 l_1} - x_{i_1 j_1 k_1 l_1}}\right).$$

Here $E(z)$ is the nearest integer less than $z$. The process ends after no more than $(m + s - 1)n + r - 1$ steps, and at the last step

$$x_{mnrs} = \hat{a}_{mn} = \hat{b}_r = \hat{c}_{ns}.$$

An unpleasant feature of the problem under consideration, as of any multi-index problem not reducible to a transportation problem, is that integrality of all basic solutions is not guaranteed even under the condition that all components of the constraint vector are integral; although it follows from the algorithm for finding a basic solution that there exists at least one basic integral solution ([1]).

The problem under consideration can be optimized in various ways.

Consider the polyhedron $H$ generated by constraints (1). Let $x^{(\omega)} \in H$. We shall find such $\bar{x}^{(\omega)}$ and $\lambda_\omega$ that the convex combination

$$x_{ijkl} = \sum_{\omega} \lambda_\omega x_{ijkl}^{(\omega)}, \tag{4}$$

$$\sum_{\omega} \lambda_\omega = 1, \qquad \lambda_\omega \geqslant 0, \tag{5}$$

also satisfies all constraints (2). Substituting (4) into (2) and (3), we obtain the following problem: find

$$\min \sum_{\omega} \lambda_\omega q^{(\omega)} \tag{6}$$

subject to

$$\sum_{\omega} f_k^{(\omega)} \lambda_\omega = b_k, \tag{7}$$

where

$$q^{(\omega)} = \sum_{i,j,k,l} p_{ijkl} x_{ijkl}^{(\omega)}; \tag{8}$$

$$f_k^{(\omega)} = \sum_{i,j,l} x_{ijkl}^{(\omega)}. \tag{9}$$

Conditions (5) are satisfied automatically. Indeed,

$$\sum_{k=1}^{r} b_k = \sum_{k=1}^{r} \sum_{\omega} f_k^{(\omega)} \lambda_\omega = \left( \sum_{\omega} \lambda_\omega \right) \left( \sum_{j=1}^{n} \sum_{l=1}^{s} c_{jl} \right).$$

Hence (5) follows. The assertion is proved.

Consider $r$ interior points of the polyhedron $H$, whose coordinates are specified as follows: $x_{ijkl}^{(\omega)} = a_{ij} c_{jl} \big/ \sum_{i=1}^{m} a_{ij}$, if $\omega = k$; $x_{ijkl}^{(\omega)} = 0$, if $\omega \neq k$. Then

$$q^{(\omega)} = \sum_{i,j,l} p_{ij\omega l} \frac{a_{ij} c_{jl}}{\sum_{i=1}^{m} a_{ij}}; \tag{10}$$

$$f_k^{(\omega)} = \begin{cases} \sum\limits_{j=1}^{n} \sum\limits_{l=1}^{s} c_{jl}, & \text{if } \omega = k, \\ 0, & \text{if } \omega \neq k. \end{cases} \tag{11}$$

Substituting (11) into (7), we obtain

$$\lambda_k = \frac{b_k}{\sum\limits_{j=1}^{n} \sum\limits_{l=1}^{s} c_{jl}}.$$

In order that the vector $(\lambda_1, \dots, \lambda_r)$ deliver the minimum in problem (6)–(7), it is necessary and sufficient that there exist a vector of estimates $\pi = (\pi_1, \dots, \pi_r)$ for which the conditions

$$\sum_{k=1}^{r} \pi_k f_k^{(\omega)} \leqslant q^{(\omega)}$$

hold for all $\omega$, whence from (8) and (9) follow the necessity and sufficiency of the condition

$$\min \sum_{i,j,k,l} x_{ijkl}(p_{ijkl} - \pi_k) \geqslant 0,$$

where all $x_{ijkl}$ satisfy conditions (1), (2).

The latter problem reduces to solving $n$ independent transportation problems [3]:

Find

$$\min \sum_{i,l} \min_{1 \le k \le r} (p_{ijkl} - \pi_k) y_{ijl} \tag{12}$$

$$\sum_{l=1}^{s} y_{ijl} = a_{ij}, \qquad \sum_{i=1}^{m} y_{ijl} = c_{jl}. \tag{13}$$

Each of the $n$ problems (12), (13) is independent of the index $j$. If $y_{ijl}^{(0)}$ is an optimal solution of problem (12), (13), then $x_{ijk}^{(0)} = y_{ijl}^{(0)}$ if $k = k^*$, and $x_{ijkl}^{(0)} = 0$ if $k \neq k^*$, where $k^*$ is defined as the minimum value of the index $k$ for which

$$\min_{1 \le k \le r} (p_{ijkl} - \pi_k) = p_{ijk^*l} - \pi_{k^*}.$$

The vector of estimates is $\pi = q_B B^{-1}$, where $q_B$ is the vector of basic "prices," and $B$ is the matrix of basic columns.

Since the initial values are

$$B = \left(\sum_{j=1}^{n}\sum_{l=1}^{s} c_{jl}\right) E$$

and

$$q_B = (q^{(1)}, \dots, q^{(r)}),$$

we have

$$B^{-1} = \left(\sum_{j=1}^{n}\sum_{l=1}^{s} c_{jl}\right)^{-1} E$$

and

$$\pi^{(k)} = \left(\sum_{j=1}^{n}\sum_{l=1}^{s} c_{jl}\right)^{-1} q^{(k)}$$

(see (10)).

Subsequently, the solution algorithm coincides completely with the standard procedure of block programming [4].

Since all the initial information for the block algorithm is obtained directly from the initial conditions, first, the amount of computation is substantially reduced (the number of block iterations of the problem), and second, it is advisable to use the multiplicative form of representing the inverse matrix, since its use is the more efficient the smaller the number of iterations [5].

Central Economic-Mathematical Institute
Academy of Sciences of the USSR

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*Note: Figure translations are in progress. See original paper for figures.*