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Abstract

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WREATH PRODUCTS AND VARIETIES OF GROUPS

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Let X be a free group of countable rank with free generators $x_1, x_2, \ldots$, and let V be a certain set of words in these generators. In an arbitrary group G, its verbal V-subgroup $V(G)$ is defined as the subgroup generated in G by all possible elements of the form $f(g_1, \ldots, g_n)$, where $f(x_1, \ldots, x_n) \in V$, $g_1, \ldots, g_n \in G$. If $V(G) = E$, then one says that the set V of identical relations is satisfied in the group G (more precisely, the elements of V are the left-hand sides of identical relations).

The variety of groups corresponding to the set V is the class of all possible groups G such that $V(G) = E$. The set V, generally speaking, is not the complete set of identical relations of the variety defined by it—one must also take into account the consequences of the identities V. The complete set of identical relations of this variety is $V(X)$.

On the set of varieties, H. Neumann ([1]) defined a natural multiplication operation: the product $\mathfrak{A}\mathfrak{B}$ of varieties $\mathfrak{A}$ and $\mathfrak{B}$ is the variety consisting of all possible extensions of groups from $\mathfrak{A}$ by means of groups from $\mathfrak{B}$. If V and W are systems of identical relations of the varieties $\mathfrak{A}$ and $\mathfrak{B}$, respectively, then a complete system of identities of the variety $\mathfrak{A}\mathfrak{B}$ is $V(W(X))$. The operation of multiplication of varieties is associative. Every variety can be represented in a unique way as a product of indecomposable varieties (see ([2,3])).

Let $\mathfrak{A}$ be a variety with system of identities V; let G be a certain group from $\mathfrak{A}$, possessing a system of generators $\Gamma = \langle g_\alpha,\ \alpha \in M \rangle$ such that every single-valued mapping of the set Γ into any group $H \in \mathfrak{A}$ extends to a homomorphism $G \to H$. Such a group G is called a **free group** of the variety $\mathfrak{A}$, or an $\mathfrak{A}$-free group, and the set Γ a system of ($\mathfrak{A}$-) free generators of the group G. The group G is isomorphic to the factor group $F/V(F)$ of an (absolutely) free group F with free generators $y_\alpha,\ \alpha \in M$. The cardinality of the set of free generators of the group G is its invariant and is called the **rank** of this group.

It is clear that the study of varieties is equivalent to the study of identical relations, and also to the study of free groups of varieties (it is known that a variety is determined by its free group of countable rank).

Products of varieties have been studied in a number of works ([1–10]). Many results in this direction are due to the use of the construction of the (discrete and complete) wreath product of groups. This is largely explained by the theorem of L. A. Kaluzhnin and M. Krasner ([11]) stating that every extension of a group A by means of B is embedded in their complete wreath product $A \operatorname{Wr} B$. In particular, since every $\mathfrak{A}\mathfrak{B}$-free group G is an extension of the $\mathfrak{A}$-free group $W(G)$ by means of the $\mathfrak{B}$-free group $G/W(G)$, the group G is embedded in $W(G) \operatorname{Wr}(G/W(G))$.

However, the method of studying products of varieties based on the indicated embedding has a number of shortcomings. First, into the group $W(G) \operatorname{Wr}(G/W(G))$ there is embedded every extension of $W(G)$ by means of $G/W(G)$; therefore the indicated wreath product is very far from the group G. Sec-

secondly, a certain inconvenience is presented by the circumstance that representatives of adjacent classes $G/W'(G)$ participate in the very definition of this embedding.

We have already noted (see ([8], [9])) that the wreath-product construction admits a broad generalization, and that some theorems show the naturalness (if not the necessity) of using such generalized wreath products in the theory of varieties.

We mean the verbal wreath product. It is based on the construction of the verbal product (see ([12])). Let V be some set of words in X. The **verbal V-product** of a set of groups $G_\alpha,\ \alpha \in M$ (denoted by $\prod\limits_{\alpha\in M}^{V} G_\alpha$) is the factor group

$$F/(V(F) \cap [G_\alpha]^F)$$

of their free product

$$F = \prod_{\alpha}^{*} G_\alpha,$$

where $[G_\alpha]^F$ is the normalized mutual commutator of the subgroups G_α in F, i.e. the normal divisor generated by all commutators

$$[g_\alpha, g_\beta] = g_\alpha^{-1} g_\beta^{-1} g_\alpha g_\beta, \quad g_\alpha \in G_\alpha,\ g_\beta \in G_\beta,\ \alpha, \beta \in M,\ \alpha \neq \beta.$$

The verbal V-product in the variety $\mathfrak{A}$, defined by the identities V, is a free operation; in particular, the V-product of $\mathfrak{A}$-free groups is an $\mathfrak{A}$-free group.

Let A and B be arbitrary groups; let, further, $A(b),\ b \in B$, be groups isomorphic to A (with isomorphisms $a \to a(b),\ a \in A$). Denote

$$K = \prod_{b\in B}^{V} A(b).$$

Take the split extension of the group K by means of B, in which the elements of B induce automorphisms in K according to the rule

$$b^{-1}a(b_1)b = a(b_1 b)$$

(it is clear that the correspondence so defined extends to an automorphism of the group K). The constructed group—we shall denote it by $A \operatorname{wr}_V B$—will be called the **verbal V-wreath product** of the group A with the group B. In the particular case when V consists of the commutator $[x_1, x_2]$, the V-product coincides with the direct product, and the V-wreath product with the ordinary (discrete) wreath product.

We list some properties of V-wreath products, some of which are obvious, while others require certain considerations. Here we shall denote by H^G the normal divisor generated by the subgroup H in the group G, and $G = A \operatorname{wr}_V B$.

a) G is generated by the subgroups $A(1)$ and B, and $A^G = K$.

b) If N is a normal divisor in A, then

$$G/N(1)^G \cong (A/N) \operatorname{wr}_V B.$$

c) If $V(A) = E$ and N is a normal divisor in B, then

$$G/N^G \cong A \operatorname{wr}_V (B/N).$$

d) If $C \subseteq B$, then

$$\{A(1), C\} \cong A \operatorname{wr}_V C.$$

e) If D is a semigroup multiplier in A, then

$$\{D(1), B\} \cong D \operatorname{wr}_V B.$$

f) If $A \in \mathfrak{A}$, $B \in \mathfrak{B}$, then

$$A \operatorname{wr}_V B \in \mathfrak{A}\mathfrak{B},$$

where V are the identities of the variety $\mathfrak{A}$.

g) If $V \subseteq W$, then there exists a homomorphism

$$A \operatorname{wr}_V B \to A \operatorname{wr}_W B.$$

For the particular case of ordinary wreath products these properties were established in ([13]).

The proposed method of studying products of varieties is based on the following theorem:

Theorem 1. Let F be an (absolutely) free group with free generators y_α, $\alpha \in M$; let N be its normal divisor; let V be some set of words; let $\mathfrak{A}$ be the variety corresponding to the identities V; and let A be an $\mathfrak{A}$-free group with free generators $a_\alpha,\ \alpha \in M$.

The mapping

$$y_\alpha \to \tilde{y}_\alpha a_\alpha (1),$$

where

$$\tilde{y}_\alpha = y_\alpha N \in F/N, \quad \alpha \in M,$$

extends to an isomorphism of $F/V(N)$ into

$$A \operatorname{wr}_V (F/N).$$

In the case where $N = W(F)$ and $\mathfrak{B}$ is the variety corresponding to W, we have

$$F/V(W(F)) = F/V(N),$$

and the indicated factor group is an $\mathfrak{A}\mathfrak{B}$-free group. The more general situation covered by Theorem 1 gives us the possibility of generalizing some known theorems.

In the particular case when $V(N) = [N, N]$, this theorem was proved by M. Hall (see ([14]), p. 256) in other terms.

On the basis of the embedding theorem just given, some approximation theorems of G. Baumslag can be reproved (we note that special cases of these theorems, when $V(N) = [N, N]$, were proved, using M. Hall' s theorem, i.e. on the basis of the same considerations as ours, by D. M. Smirnov ([15])). For example:

Theorem 2. *If (under the hypotheses of Theorem 1) F/N and $N/V(N)$ are approximated by finite groups (finite p-groups), then $F/V(N)$ is approximated by finite groups (respectively, finite p-groups).*

Suppose that under the hypotheses of Theorem 1, F has a finite set of generators; N (as a normal divisor) is generated by a recursively enumerable set of elements from F; V is a recursively enumerable set of words in X. Then the following is true.

Theorem 3. *If there are algorithms for the word problem in F/N and in $\mathfrak{A}$-free groups of finite ranks, then there exists an algorithm for the word problem in $F/V(N)$.*

In particular, there is always an algorithm for the word problem in F/N_k (where N_k is the k-th term of the lower central series of the group N), if there exists an algorithm for the word problem in F/N.

We shall say that a variety $\mathfrak{A}$ is generated by a group G if it is the smallest variety containing G.

P. M. Neumann (see ([5])), generalizing a theorem of G. Higman, proved that if a variety $\mathfrak{B}$ contains non-Abelian groups and the variety $\mathfrak{A}$ is nontrivial, then $\mathfrak{A}\mathfrak{B}$ is not generated by any finite group. In connection with this there arises the general question: when can $\mathfrak{A}\mathfrak{B}$ be generated by a finite group? This problem is evidently equivalent to the following: when does a variety generated by a finite group decompose into a product of varieties? The a priori answer—never—is refuted by the symmetric group S_3 of degree three.

Theorem 4. *The product $\mathfrak{A}\mathfrak{B}$ of varieties $\mathfrak{A}$ and $\mathfrak{B}$ is generated by a finite group if and only if:*

a) *the varieties $\mathfrak{A}$ and $\mathfrak{B}$ have finite relatively prime exponents;*

b) *all groups in $\mathfrak{B}$ are Abelian, and all groups in $\mathfrak{A}$ are nilpotent.*

The necessity of the conditions is proved with the aid of wreath products on the basis of Theorem 1. The sufficiency follows from a theorem of Cross (see ([16])).

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Note: Figure translations are in progress. See original paper for figures.