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Abstract

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A NEW GENERAL METHOD FOR INVESTIGATING LINEAR OPERATOR EQUATIONS OF THE TYPE OF SINGULAR INTEGRAL EQUATIONS

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The paper gives a method for investigating the Noether property* of linear (bounded) operators of "local type," acting from $\mathcal{L}_p(X)$ into $\mathcal{L}_p(X)$, where X is a bicompact Hausdorff space of finite dimension with measure. In application this method gives necessary and sufficient conditions for the Noether property for singular integrals: 1) with continuous symbol in the class $\mathcal{L}_p$ $(1 < p < \infty)$, under more general assumptions than those of S. G. Mikhlin ([1,2]); 2) for previously uninvestigated integrals with piecewise-continuous symbol, a special case of which is integrals over a manifold with boundary. The latter investigation is carried out in the class $\mathcal{L}_2$. Here we give the general method and a small part of application 1), illustrating the general method.

1°. **Operators of local type.** We introduce the necessary notation. Let X be a bicompact Hausdorff space of finite dimension with a nonnegative (possibly infinite) measure μ, defined on some σ-ring of subsets of X. We shall assume that this σ-ring contains all open and closed sets**. If all these conditions are fulfilled, we shall briefly write that X is a b.h.m. Everywhere below we shall assume that X is a b.h.m. An example of a b.h.m. is extended Euclidean space with Lebesgue measure.

The space of measurable (complex- or real-valued) functions summable to the power p $(1 \le p < \infty)$ will be denoted by $\mathcal{L}_p(X)$. The space of vectors $f = (f_1, f_2, \dots, f_n)$ with components from $\mathcal{L}_p(X)$ will be denoted by $\mathcal{L}_p^n(X)$. The norm in $\mathcal{L}_p^n(X)$ is introduced by the equality

$$\|f\| = \sum_{k=1}^{n} \|f_i\|.$$

Let A be a linear operator. The quantity $\inf_x \|A-K\|$, where K runs through the set of completely continuous operators, will be denoted by $\|\!|A|\!\|$. If $\|\!|A_1-A_2|\!\| = 0$, then we shall say that A_1 is equivalent to A_2, and write $A_1 \sim A_2$.

If M is a measurable set, then P_M will denote the operator acting according to the rule:

$$P_M f(x) = f(x) \quad \text{for } x \in M; \qquad P_M f(x) = 0 \quad \text{for } x \notin M.$$

Definition 1. An operator $A(\mathcal{L}_p^n(X) \to \mathcal{L}_p^n(X))$ is called an operator of local type if, for any two closed nonintersecting sets F_1 and F_2, the operator $P_{F_1} A P_{F_2}$ is completely continuous.

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* A linear operator B is called a Noether operator if: 1) its range is closed; 2) the number $\alpha(B)$ of linearly independent solutions of the homogeneous equation $Bf = 0$ is finite; 3) the number $\beta(B) = \alpha(B^*)$, where B^* is the adjoint of B, is also finite.

** For the concepts of σ-ring, measure, and integral see ([5]); topological concepts (dimension, bicompactness, etc.) are contained in ([6]).

2°. The concept of a local regularizer and local Noetherian property.

Definition 2. We shall say that the operator R_l (R_r) is a **local left** (**right**) **regularizer** of the operator A ($\mathcal{L}_p^n(X) \to \mathcal{L}_p^n(X)$) at the point $x_0(\in X)$, if there exists a neighborhood u ($\ni x_0$) such that

$$R_l A P_u \sim P_u \qquad (P_u A R_r \sim P_u).$$

For brevity, instead of "local left (right) regularizer at the point x_0" we shall write $l.x_0$ ($r.x_0$).

Definition 3. The operator A will be called a **local Noether operator** at the point $x_0(\in X)$, if it possesses left and right local regularizers at the point x_0.

The main results of the proposed method are the following:

Theorem 1. *In order that an operator of local type A, acting from $\mathcal{L}_p^n(X)$ into $\mathcal{L}_p^n(X)$, be a Noether operator, it is necessary and sufficient that at each point $x(\in H)$ the operator A be a local Noether operator.*

Theorem 2. *If an operator A of local type, acting from $\mathcal{L}_p^n(X)$ into $\mathcal{L}_p^n(X)$, at each point $x(\in X)$ has a right (left) local regularizer of local type, then the operator has a right (left) regularizer and, consequently, the range of the operator A is closed and $\beta(A) < \infty$ ($\alpha(A) < \infty$).*

Thus the problem of investigating the operator A as a whole is reduced to a local investigation of the operator. The following concepts and theorems are intended to make the local investigation effective.

3°. Equivalence at a point.

Definition 4. Operators A, B ($\mathcal{L}_p^n(X) \to \mathcal{L}_p^n(X)$) will be called **equivalent at the point** $x_0(\in X)$, if for every $\varepsilon(> 0)$ there exists a neighborhood u of the point x_0 such that $\|(A-B)P_u\| < \varepsilon$. In abbreviated form we shall write this fact as $A \overset{x_0}{\sim} B$. To illustrate this concept we give two examples.

Example 1. Multiplication by a continuous function $a(x)$ is equivalent at the point x_0 to multiplication by the constant $a(x_0)$.

Example 2. Consider the singular integral

$$Af = \int_{E_m} \frac{\Omega(x, x-y)}{|x-y|^m} f(y)\, dy, \tag{1}$$

where $\Omega(x,t)$ is a measurable function of both variables, homogeneous of degree 0 in t; E_m is the extended Euclidean space. A. Calderón and A. Zygmund ([4]) proved that the operator A will be a linear (bounded) operator acting from $\mathcal{L}_p(E_m)$ into $\mathcal{L}_p(E_m)$ ($p > 1$), if the following conditions are satisfied:

$$\text{a)}\ \int_S \Omega(x,\theta)\, d\theta = 0 \quad (S \text{ is the unit sphere});$$

$$\text{b)}\ \operatorname*{vrai\,max}_x \int_S |\Omega(x,\theta)|^{p'}\, d\theta < \infty \quad \left(\frac{1}{p} + \frac{1}{p'} = 1\right).$$

In this case

$$\|A\|_p \leqslant c_p \left[\operatorname*{vrai\,max}_x \int_S |\Omega(x,\theta)|^{p'}\, d\theta\right]^{1/p'}, \tag{2}$$

where c_p depends only on p. We shall assume that these conditions are fulfilled. The operator A is an operator of local type. Under the additional

condition

$$)\quad \lim_{x\to x_0} \int_S |\Omega(x,\theta) - \Omega'(x_0,\theta)|^{p'}\, d\theta$$

from estimate (2) the equivalence follows without difficulty:

$$A \overset{x_0}{\sim} A_{x_0}, \tag{3}$$

where

$$A_{x_0} f = \int_{E_m} \frac{\Omega(x_0, x-y)}{|x-y|^m} f(y)\, dy, \quad x_0 \in E_m$$

(possibly $x_0 = \infty$).

Theorem 3. If A, B are operators acting from $\mathcal{L}_p^n(X)$ into $\mathcal{L}_p^n(X)$, and the operator A is equivalent at the point x_0 to the operator B, then the operators A, B simultaneously either possess an l.s. ($n.s.$) at x_0, or do not possess one. If, in addition, the operators A, B are of local type, then they simultaneously either possess an l.s. ($n.s.$) of local type at x_0, or do not possess one.

A simple consequence of Theorem 3 is

Theorem 4. Under the conditions of the preceding theorem, the operators A, B simultaneously either are local Noether operators at the point x_0, or are not such operators.

Theorems 3 and 4 make it possible, instead of carrying out a local investigation of the original operator, to investigate simpler operators equivalent to the original one at the given point. For general singular integrals such simpler operators, as we have seen, are singular integrals with a kernel depending only on the difference.

4°. The concept of quasi-equivalence. A generalization of the concept of equivalence at a point to the case of operators acting in different spaces turns out to be very useful.

Let X, Y be two b.x.m. A homeomorphic measurable transformation φ of some neighborhood $u(\subset X)$ onto a neighborhood $v(\subset Y)$ will be called **measure-nondistorting** if

$$c_1\mu(E) \leq \eta[\varphi(E)] \leq c_2\mu(E),$$

where $c_1, c_2 (> 0)$ are constants; E is any measurable part of u; μ, η are the measures on X, Y, respectively. By T_φ we shall denote the operator which assigns to every function $f \in \mathcal{L}_p^n(Y)$ vanishing outside v the function $T_\varphi f = f[\varphi(x)]$, when $x \in u$, and 0, when $x \notin u$.

Definition 5. We shall say that the operator $A(\mathcal{L}_p^n(X) \to \mathcal{L}_p^n(X))$ at the point $x_0(\in X)$ is **quasi-equivalent to the operator** $B(\mathcal{L}_p^n(Y) \to \mathcal{L}_p^n(Y))$ at the point $y_0(\in Y)$, if there exist two neighborhoods $(x_0 \in) u(\subset X)$, $(y_0 \in) v(\subset Y)$ and a measure-nondistorting transformation φ of u onto v, such that $y_0 = \varphi(x_0)$ and

$$T_{\varphi^{-1}} P_u A P_u T_\varphi P_v \overset{y_0}{\sim} P_v B P_v.$$

In abbreviated form we shall write this fact as $A \overset{x_0\, y_0}{\sim} B$. If it is necessary to indicate the transformation φ, we shall say that A at the point x is φ-equivalent to the operator B at the point y_0 and write $A \overset{x_0\, y_0}{\sim} B$.

Theorem 5. If the operators $A(\mathcal{L}_p^n(X) \to \mathcal{L}_p^n(X))$ and $B(\mathcal{L}_p^n(Y) \to \mathcal{L}_p^n(Y))$ are of local type, then Theorems 3 and 4 remain valid when the concept of equivalence is replaced by the concept of quasi-equivalence.

5°. The concept of an enveloping operator. Suppose we are given a family of operators $A_x(\mathcal{L}_p^n(X) \to \mathcal{L}_p^n(X))$, depending on $x(\in X)$.

Definition 6. An operator $A(\mathcal{L}_p^n(X) \to \mathcal{L}_p^n(X))$ will be called **enveloping** for the family A_x, if at every point $x(\in X)$ the equivalence

$$A \overset{x}{\sim} A_x$$

holds.

Definition 7. The family A_x will be called **locally continuous** if for each point $x_0 \in X$ and for every $\varepsilon(> 0)$ there exists a neighborhood $u(\ni x_0)$ such that $\|(A_{x_0} - A_x)P_u\| < \varepsilon$ whenever $x \in u$.

Theorem 6. *Let A_x be a locally continuous family of operators of local type. Then there exists an enveloping operator of local type, unique up to an entirely continuous summand.*

6°. Application to singular integrals. Consider the singular integral

$$Af = a(x)f(x) + \int_{E_m} \frac{\Omega(x, x-y)}{|x-y|^m} f(y)\, dy, \tag{4}$$

where $a(x)$ is a square matrix depending continuously on x; Ω is also a matrix, the entries of which satisfy conditions a), b), and condition c) at every point $x_0 \in E_m$. Then the following holds.

Theorem 7. *The operator $A(\mathcal{L}_p^n(E_m) \to \mathcal{L}_p^n(E_m))$ is a Noether operator if and only if the determinant of the symbol matrix $\Phi_A(x, \xi)$ (in the sense of S. G. Mikhlin [1]) does not vanish for any $\xi \in S$ and $x \in E_m$.*

This theorem generalizes the results of S. G. Mikhlin, who gave sufficient conditions for Noetherianity for a narrower class of operators A (see [1], theorems (2.36), (1.26), (1.33)), and of I. Ts. Gohberg [3], who established the necessity of these conditions for $p = 2$.

This generalization became possible because a significant part of the difficulties that had previously been overcome by analytic methods is resolved by Theorems 1 and 4.

It should also be noted that the regularizer of the operator (4) may itself have the form (4), but it may be constructed on the basis of Theorem 6 on the enveloping operator. One may introduce a class of operators more general than the class of integrals of the form (4), in which the phenomenon just noted is not observed.

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Note: Figure translations are in progress. See original paper for figures.