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Abstract

Full Text

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ON INFINITE PRODUCTS OF LEBESGUE SPACES

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Lebesgue spaces considered in this paper will be assumed to have no atoms, i.e., to be isomorphic to the unit interval with the usual Lebesgue measure on it.

Let a family $\{E_t\}$ of Lebesgue spaces be given, where t ranges over a set T of cardinality τ. Denote by E^τ the space whose points are all possible systems (x_t), $x_t \in E_t$. For a measurable set $A \in E_{t'}$, let $\bar{A} = \{(x_t) : x_{t'} \in A\}$, and let $S_{t'}$ be the σ-algebra of sets $\bar{A}$ for $A \in E_{t'}$. Suppose that, for any $t_1, t_2, \dots, t_n$, on the smallest σ-algebra containing the σ-algebras $S_{t_1}, S_{t_2}, \dots, S_{t_n}$, a measure is given such that for $A \in S_{t_i}$, $m\bar{A} = mA$. If all these measures are consistent (see (1)), then they extend to a measure m, defined on the smallest σ-algebra Q containing all σ-algebras S_t, $t \in T$. Completing this measure in the usual way, we obtain a measure space, which we shall call the **product of the spaces** E_t. If for any $t_1, t_2, \dots, t_n$ and measurable sets $A_1 \subset E_{t_1}, A_2 \subset E_{t_2}, \dots, A_n \subset E_{t_n}$

$$m\left(\bigcap_{i=1}^{n} \bar{A}_i\right) = \prod_{i=1}^{n} mA_i,$$

then the product of the spaces E_t will be called the **direct product** and will be denoted by Ω^τ.

The **metric weight of a measure space** E is the least cardinal number τ for which there exists a set $\mathfrak{B}$ of cardinality τ of measurable subsets of E such that, for every measurable set $A \subset E$, there exists a countable sequence $A_1, A_2, \dots, A_n, \dots$ of sets from $\mathfrak{B}$ for which

$$m((A \cap \bar{A}_n) \cup (\bar{A} \cap A_n)) \to 0.$$

A space E is called **homogeneous** if each of its measurable subsets of positive measure, considered as a measure space, has metric weight equal to the metric weight of the space E.

D. Maharam ([2]) showed that the metric structure of a homogeneous space of metric weight τ is isomorphic to the metric structure of the space Ω^τ. In connection with this there arises the problem of describing the class of measure spaces for which not only their metric structures but the spaces themselves are isomorphic to the space Ω^τ. A partial solution of this problem is given by the following theorem.

Theorem 1. *Every product E^τ of Lebesgue spaces which is a homogeneous space with metric weight τ is isomorphic to the direct product Ω^τ.*

Let, for the σ-algebra S, $\zeta(S)$ be a partition, each element of which either is contained in, or has empty intersection with, any set from S, and for any two elements of $\zeta(S)$ there is a set from S containing one element and not containing the other.

A σ-algebra S is called **Lebesgue** if the quotient space with respect to the partition $\xi(S)$ is a Lebesgue space. For a certain collection of σ-algebras S_γ we denote by $[S_\gamma]$ the σ-algebra consisting of all sets from Q that are sums of elements of the partition $\prod_\gamma \xi(S_\gamma)$. Two σ-algebras S_1 and S_2 are called **independent** if, for any $A \in S_1$ and $B \in S_2$, $m(A \cap B) = mA \cdot mB$.

In the proof of Theorem 1 the following two lemmas are used.

Lemma 1. *Let a set $T' \subset T$ have cardinality less than τ, and let $R \subset T \setminus T'$ be a countable set.*

There exists a Lebesgue σ-algebra $S \subset Q$, independent of $[S_t,\, t \in T']$, and such that

$$[S_t,\, t \in R] \subset [S,\, S_t,\, t \in T'].$$

The proof of this lemma is a slight modification of the proof of Theorem No. 3, § 4, from the paper ([3]).

Lemma 2. *Let ξ_1 and ξ_2 be two measurable partitions of a Lebesgue space whose product is the partition into points, and suppose that every element of the partition ξ_1 is a set of cardinality continuum.*

Then there exists a measurable partition $\xi_2' = \xi_2 \pmod 0$ such that the product $\xi_1 \times \xi_2'$ is the partition into points and every element of ξ_1 has nonempty intersection with every element of ξ_2'.

To prove Theorem 1, arrange the set T into a transfinite sequence

$$t_1, t_2, \dots, t_r, \dots,$$

where r runs through the set of transfinite numbers less than ω_τ—the first transfinite number of cardinality τ, and put $S' = S_{t_1}$.

Suppose that, for some $\alpha < \omega_\tau$, Lebesgue σ-algebras $S^r \subset Q$, $1 \le r \le \alpha - 1$, have been constructed with the following properties:

1) For any two groups of indices $r_1, \dots, r_n$ and $r', \dots, r^k$, the σ-algebras

$$[S^{r_1}, \dots, S^{r_n}] \quad \text{and} \quad [S^{r'}, \dots, S^{r^k}]$$

are independent.

2) There exists a set $T_{\alpha-1} \subset T$, whose cardinality is equal to the cardinality of $\alpha - 1$, containing all t_r, $r \leq \alpha - 1$, and such that

$$Q(\alpha-1) = [S^r,\, r \leq \alpha - 1] = [S^t,\, t \in T_{\alpha-1}].$$

3) Every intersection

$$\bigcap_{1 \leq r \leq \alpha-1} C_r,$$

where C_r is an element of the partition $\xi(S^r)$, is nonempty.

Using Lemma 1, construct countable sequences of Lebesgue σ-algebras

$$S_1, S_2, \dots, S_n, \dots, \qquad S_n \in Q,$$

and countable sets

$$R_0 = (t_\alpha), \quad R_1 \subset T \setminus (T_{\alpha-1} \cup R_0), \dots, \quad R_n \subset T \setminus \left(T_{\alpha-1} \cup \left(\bigcup_{i=0}^{n} R_i \right) \right), \dots$$

such that

$$[Q(\alpha-1), S_1] \supset S_{t_\alpha}, \qquad [Q(\alpha-1), S_1, \dots, S_n] \supset [S_t,\, t \in R_{n-1}],$$

$$S_n \subset \left[Q(\alpha-1), S_t,\, t \in \bigcup_{i=1}^{n} R_i \right]$$

and S_n is independent of $[Q(\alpha-1), S_1, \dots, S_{n-1}]$.

(If $t_\alpha \in T_{\alpha-1}$, then R_0 is empty, and S_1 is any Lebesgue σ-algebra independent of $Q(\alpha-1)$.) Let

$$S_\alpha = [S_1, S_2, \dots, S_n, \dots],$$

$$T_\alpha = T_{\alpha-1} \cup \left(\bigcup_{n=0}^{\infty} R_n \right),$$

and let $R \subset T_{\alpha-1}$ be such a countable set that

$$[S_t,\, t \in R \cup (\bigcup_{n=0}^{\infty} R_n)] \supset S_\alpha.$$

Let $S = [S_t,\, t \in R]$.

The quotient space with respect to the partition $\xi(S) \times \xi(S_\alpha)$ is a Lebesgue space, and the partitions $\xi(S)$ and $\xi(S_\alpha)$ induce in it partitions ξ_1 and ξ_2,

satisfying the conditions of Lemma 2. Let ξ be the partition of the space E^τ induced by the partition ξ_2', whose existence is asserted by this lemma. Denote by S^α some σ-algebra of sets from Q for which $\xi(S^\alpha) = \xi$. The S^α, together with the previously constructed S^r, satisfy conditions 1), 2), and 3). If, however,

α is a limit ordinal and the S^r have been constructed for all $r < \alpha$, then S^α is constructed in the same way as above, but instead of $Q(\alpha - 1)$ and $T_{\alpha-1}$ one takes $Q(\alpha-) = [S^r,\ r < \alpha]$ and $T_{\alpha-} = \bigcup_{r<\alpha} T^r$. After the completion of this process the isomorphism between E^τ and Ω^τ is obvious.

Let E be a topological space on which a nonatomic measure m is given, satisfying the following conditions:

1. $mE = 1$.
2. The measure space E is homogeneous.
3. There exists a system Σ of measurable sets, whose cardinality is equal to the metric weight of the space E, such that every measurable set A has the form $A = B \cup A_0,\ A_0 \subset B_0$, where B and B_0 belong to the smallest σ-algebra containing all the sets of the system Σ, $mB_0 = 0$, and for any two points of E there is a set in Σ containing one point and not containing the other.

All these conditions are necessary in order that the space E be isomorphic to Ω^τ. The following question seems interesting: how must the measure m be related to the topology of E in order that the space E be isomorphic to Ω^τ? Let us write down three more conditions.

4. For every measurable set A,

$$mA = \sup_{F \subset A} mF,$$

where F ranges over measurable bicompact sets contained in A.

Let Q be the smallest σ-algebra containing all measurable bicompact sets.

5. Q contains a system Σ possessing property 3.
6. For any Lebesgue σ-algebra $Q' \subset Q$ there is a Lebesgue σ-algebra $Q'' \supset Q'$, $Q'' \subset Q$, such that all elements of the partition $\xi(Q'')$ are bicompact sets and each element of the partition $\xi(Q')$ contains at least two elements of the partition $\xi(Q'')$.

With the help of Theorem 1 one can prove the following theorem.

Theorem 2. If a topological space with measure has metric weight continuum and satisfies conditions 1-6, then it is isomorphic to the space Ω^τ, where τ is the continuum.

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Note: Figure translations are in progress. See original paper for figures.