Mathematics

V. G. SPRINDZHUK

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Abstract

Full Text

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V. G. SPRINDZHUK

ON MAHLER' S HYPOTHESIS

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Let ω be a transcendental number. For a given positive integer n, define $w_n(\omega)$ as the exact upper bound of those numbers w for which there exist infinitely many integral polynomials P of degree not exceeding n satisfying the inequality

$$|P(\omega)| < h^{-w} \quad (h \to \infty), \tag{1}$$

where h is the height of the polynomial P. Put

$$\frac{1}{n} w_n(\omega) = \begin{cases} \theta_n(\omega), & \text{if } \omega \text{ is real,} \\ \eta_n(\omega), & \text{if } \omega \text{ is complex.} \end{cases}$$

It is well known ([4], p. 96) that

$$\theta_n(\omega) \geqslant 1, \quad \eta_n(\omega) \geqslant \frac{1}{2} - \frac{1}{2n} \quad (n = 1, 2, ...) \tag{2}$$

for all transcendental numbers ω. On the other hand, K. Mahler proved ([1]) that for almost all (in the sense of Lebesgue measure) real and complex numbers

$$\theta_n(\omega) \leqslant 4, \quad \eta_n(\omega) \geqslant \frac{7}{2} \quad (n = 1, 2, ...), \tag{3}$$

and conjectured that in the last inequalities one may take 1 and 1/2, respectively, instead of 4 and 7/2. Thus, according to Mahler' s hypothesis and by virtue of (2),

$$\sup_{(n)} \theta_n(\omega) = 1, \quad \sup_{(n)} \eta_n(\omega) = \frac{1}{2} \quad (n = 1, 2, ...)$$

for almost all numbers ω.

Koksma proved ([2]) that Mahler' s constants 4 and 7/2 can be replaced by 3 and 5/2, respectively; then LeVeque ([3]) obtained an analogous result for the constants 2 and 3/2. Finally, B. Folkman ([7]) obtained the inequalities

$$\theta_n(\omega) \leqslant \frac{3}{2}, \quad \eta_n \leqslant \frac{3}{4} - \frac{1}{2n} \quad (n = 1, 2, ...)$$

for almost all numbers.

The author of the present paper proved ([11]) the existence of such numbers θ_n, η_n that, for almost all ω,

$$\theta_n(\omega) = \theta_n, \quad \eta_n(\omega) = \eta_n \quad (n = 1, 2, ...)$$

and obtained ([12]) the inequalities

$$\theta_n < \frac{4}{3}, \quad \eta_n < \frac{2}{3} \quad (n = 1, 2, ...).$$

In view of the results of Kubilius ([10]), Folkman ([5,6]), and Cassels ([8]), the equalities

$$\theta_2 = \theta_3 = 1, \quad \eta_2 = \frac{1}{4}, \quad \eta_3 = \frac{1}{3}. \tag{4}$$

hold.

In this paper we briefly set forth the scheme of the proof of Mahler' s hypothesis for complex numbers, i.e., the scheme of the proof of the inequalities

$$\frac{1}{2} - \frac{1}{2n} \leqslant \eta_n \leqslant \frac{1}{2} \quad (n = 1, 2, ...). \tag{5}$$

Analogously, the inequalities

$$1 \ll \theta_n \ll 1 + \frac{1}{n!} \qquad (n = 1, 2, ...)$$

are proved. Similar results are valid for fields of p-adic numbers and formal power series (for an introduction see [9, 13]). The method of reasoning is based on the ideas and results of the author' s papers [11, 12].

Lemma 1. Let P be a polynomial of degree n, height h, without multiple roots $\chi_1, \chi_2, \dots, \chi_n$; let ω be a complex number,

$$|\omega - \chi_1| = \min_{(i)} |\omega - \chi_i| \qquad (i = 1, 2, \dots, n).$$

Then, if $|P(\omega)| < h^{-w}$, then

$$|\omega-\chi_1|^2 < c(n, \operatorname{Im}\omega)\, h^{-5/3-\frac{4w-n}{3}}\, |D(P)|^{-1/6},$$

where $D(P)$ is the discriminant of P.

For the proof see [12].

Lemma 2. Let Δ be a measurable set in the plane, $\operatorname{mes}\Delta < \varepsilon$.

Let a system $\Lambda = \bigcup_{i=1}^{\infty} \lambda_i$ of bounded simply connected domains λ_i be given with the conditions

$$\operatorname{mes}(\lambda_i \cap \Delta) \geq \frac{1}{2} \operatorname{mes}\lambda_i, \qquad \operatorname{mes}\lambda_i > c d_i^2 \qquad (i = 1, 2, \dots),$$

where d_i is the diameter of the set λ_i, $c > 0$ is a constant. Then $\operatorname{mes}\Lambda < c_1\varepsilon$, where c_1 is a constant.

The proof of this lemma is elementary, and we omit it.

We pass to the proof of the right-hand side of inequality (5). It is enough to consider the set $\mathcal{P}_n(h)$ of irreducible polynomials $P = a_0 + a_1 x + \dots + a_{nx}^n$ satisfying

$$\max(|a_0|, \dots, |a_{n-1}|) \ll a_n = h$$

and to assume that $\omega \in \Omega$, where Ω is a bounded domain in the upper half-plane [12].

Let $\sigma(P)$ be the set of all ω satisfying (1) with some $w = w_0 = w_{n-1} + \delta$, $\delta > 0$, where $w_n = n\eta_n$ $(n = 2, 3, \dots)$. It is clear that $\sigma(P)$ is a system of at most n nonintersecting simply connected domains. Let $\sigma_1(P)$ be one such domain. It is possible that $\sigma_1(P)$ contains points ω_0 belonging to systems $\sigma(Q)$, $Q \in \mathcal{P}_n(h)$, $Q \neq P$. For such a point ω_0, $|R(\omega_0)| < 2h^{-w_0}$, where $R = P - Q \neq 0$ is a polynomial of degree at most $n-1$, height at most $2h$. We say that the domain $\sigma_1(P)$ is **essential** if in it the set of points ω_0 with the indicated property has measure less than $\frac{1}{2} \operatorname{mes}\sigma_1(P)$, and **inessential** in the opposite case. It can be shown that the domains $\sigma_1(P)$ have the property $\operatorname{mes}\sigma_1(P) > c(n)d^2$, where d is the diameter of the domain $\sigma_1(P)$. This is derived with the aid of Lemma 1 from the fact that under the conditions of this lemma, for $w \geq w_{n-1}$,

$$|\omega-\chi_1| \asymp \begin{cases} |P(\omega)| : |P'(\chi_1)|, & \text{if } |\omega-\chi_1| \leq 2|\chi_1-\chi_2|, \\ \left(|P(\omega)|\, |\chi_1-\chi_2| : |P'(\chi_1)|\right)^{1/2}, & \text{if } |\omega-\chi_1| > 2|\chi_1-\chi_2|, \end{cases}$$

where χ_2 is the root of P nearest to ω after χ_1. Applying Lemma 2 to the inessential domains, we find that the measure of the set of points falling into infinitely many inessential domains is zero for any $\delta > 0$.

We note that the number of polynomials P having an essential domain $\sigma_1(P)$ with the condition $\operatorname{mes}\sigma_1(P) \geq \lambda > 0$ will be $\ll \lambda^{-1}$.

The domain $\sigma_1(P)$ contains at least one root of the polynomial P, say χ_1. Let us place the remaining roots of the polynomial P lying in the upper half-

in order of proximity, $x_2, x_3, \dots, x$, so that

$$|x_1 - x_2| \leq |x_1 - x_3| \leq \dots \leq |x_1 - x_k| \qquad \left(k \leq \frac{n}{2}\right).$$

Now put $|x_1 - x_i| = h^{-\rho_i}$ $(i = 2, 3, \dots, k)$. Take an arbitrary, but subsequently fixed, $\varepsilon > 0$, put $m = [n/\varepsilon] + 1$, and define the integers r_i by the inequalities

$$\frac{r_i}{m} \leq \rho_i < \frac{r_i + 1}{m} \qquad (i = 2, 3, \dots, k).$$

Then

$$h^{1-s-\varepsilon} \ll |P'(x_1)| \ll h^{1-s}, \qquad h^{-\frac{r_i+1}{m}} < |x_1 - x_i| \leq h^{-\frac{r_i}{m}} \qquad (i = 2, 3, \dots, k),$$

where $s = \frac{1}{m}(r_1 + r_2 + \dots + r_k)$. It is easy to see that there exist at most $c(n, \varepsilon)$ different systems $(r_1, r_2, \dots, r_k)$. Next, we divide the polynomials P with the same number k into three classes depending on the three possibilities:

$$1^\circ. \quad \frac{r_2}{m} < \frac{n}{4} - \frac{s_1}{2},$$

$$2^\circ. \quad s_1 < \frac{n}{6}, \quad \frac{r_2}{m} \geq \frac{n}{4} - \frac{s_1}{2},$$

$$3^\circ. \quad s_1 \geq \frac{n}{6}.$$

Here $s_1 = s - r_2/m$. We now consider separately the polynomials of each class.

1°. Let ω_1 be the point on the boundary $\sigma_1(P)$ nearest to x_1. We have

$$|\omega_1 - x_1| \ll h^{-\frac{1}{2}(w_0+1-s-\varepsilon)}.$$

Therefore $|\omega_1 - x_1| \leq |x_1 - x_2|$, so that

$$|\omega_1 - x_1| \gg |P(\omega_1)| : |P'(x_1)| > h^{-w_0-1+s}.$$

Since the circle of radius $|\omega_1 - x_1|$ with center at x_1 lies inside $\sigma_1(P)$, it follows that

$$\operatorname{mes}\sigma_1(P) \gg h^{-2(w_0+1-s)},$$

and the number of polynomials of the first class of the given height h will be

$$\ll h^{2(w_0+1-s)}.$$

Consequently, the measure of the set of numbers ω for which (1) holds with $P \in \mathfrak{P}_n(h)$ of the first class will be

$$\ll h^{-2(w+1-s-\varepsilon)} h^{2(w_0+1-s)} = h^{-2(w-w_0)+2\varepsilon}.$$

If $w > w_0 + \frac{1}{2} + \varepsilon = w_{n-1} + \frac{1}{2} + \delta + \varepsilon$, then inequality (1) has a finite number of solutions in polynomials of the first class for almost all ω.

2°. In this case $r_3/m \le n/4 - s_1/2$. Suppose that there exists a pair of polynomials $P_1, P_2 \in \mathfrak{P}_n(h)$ satisfying

$$|x_1^{(1)} - x_1^{(2)}| < h^{-\frac{n}{4}+\frac{s_1}{2}}.$$

Since then

$$|x_i^{(1)} - x_j^{(2)}| \le 2h^{-\frac{1}{m}r_{\max(i,j)}} + h^{-\frac{n}{4}+\frac{s_1}{2}} \qquad (i,j = 1,2,\dots,k),$$

we have

$$|x_i^{(1)} - x_j^{(2)}| \ll \begin{cases} h^{-\frac{n}{4}+\frac{s_1}{2}}, & \text{if } \max(i,j) \le 2, \\ h^{-\frac{1}{m}r_{\max(i,j)}}, & \text{if } \max(i,j) \ge 3. \end{cases}$$

Consequently,

$$h \ll |R(P_1,P_2)| \ll h^{2n} h^{-8(\frac{n}{4}-\frac{s_1}{2})} \prod_{\max(i,j)\ge 3} h^{-\frac{2}{m}r\max(i,j)} \ll h^{-6s_1},$$

which is impossible. Thus, the circle with center at $\chi_1^{(1)}$ and radius $h^{-\frac{n}{4}+\frac{s_1}{2}}$ is always free of other roots $\chi_1^{(2)}$. Therefore the number of polynomials of the second class is

$$\ll h^{\frac{n}{2}-s_1}.$$

The measure of the set of numbers ω for which (1) holds with polynomials $P \in \mathfrak{P}_n(h)$ of the second class will be

$$\ll h^{-w-1+s_1+\varepsilon} h^{\frac{n}{2}-s_1} = h^{-w-1+\frac{n}{2}+\varepsilon},$$

and one should take $w > n/2 + \varepsilon$.

3°. Let, as in 1°, ω_1 be the point on the boundary $\sigma_1(P)$ nearest to χ_1. First consider those polynomials for which

$$|\omega_1 - \chi_1| \geq 2|\chi_1 - \chi_2|.$$

We have

$$|\omega_1 - \chi_1| \gg h^{-1/2(w_0+1-s_1)},$$

and the number of polynomials is

$$\ll h^{w_0+1-s_1}.$$

By Lemma 1, the measure of the set of numbers ω for which (1) holds with $P \in \mathfrak{P}_n(h)$ of the third class will be

$$\ll h^{5/3-\frac{4w-n}{3}} h^{w_0+1-s_1} \ll h^{-2/3-4/3\,w+w_0+\frac{n}{6}}.$$

Therefore we assume

$$4w > 3w_0 + \frac{n}{2} + 1.$$

The remaining polynomials of the third class satisfying

$$|\omega_1 - \chi_1| \leq 2|\chi_1 - \chi_2|$$

are considered analogously to the polynomials of the first class.

Summing up, we conclude from the preceding that

$$w_n \ll \max\left(w_{n-1} + \frac{1}{2},\ \frac{n}{2},\ \frac{3}{4}w_{n-1} + \frac{n}{8} + \frac{1}{4}\right).$$

In view of (4), it follows from this that

$$w_n \ll n/2 \qquad (n = 1, 2, \ldots).$$

We note that the above arguments concerning polynomials of the first and third classes do not prevent an attempt to obtain the equalities

$$\theta_n = 1, \qquad \eta_n = \frac{1}{2} - \frac{1}{2n}$$

for all n. By developing somewhat the reasoning concerning polynomials of the second class, one can obtain inequalities of the form

$$\theta_n \ll 1 + \frac{c_1}{n}, \qquad \eta_n \ll \frac{1}{2} - \frac{c_2}{2n} \qquad (n = 1, 2, ...)$$

with some absolute constants c_1, c_2, $0 < c_1, c_2 < 1$.

Institute of Mathematics and Computer Technology
Academy of Sciences of the BSSR

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Note: Figure translations are in progress. See original paper for figures.