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Abstract

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ON CERTAIN TYPES OF INTERFERENCE FOR ENTIRE FUNCTIONS OF FINITE DEGREE

(Presented by Academician S. N. Bernstein on November 26, 1963)

Let us denote by $B_\sigma^{(1)}$ the class of entire functions $f(z)$ $(z = x + iy)$ of degree $\leqslant \sigma$ such that

$$\left| f\left(\frac{k\pi}{\sigma}\right) \right| \leqslant M \quad (k = 0,\ \pm 1,\ \pm 2, \dots;\ M \text{ is some constant}), \tag{1}$$

$$\lim_{x \to \pm\infty} \frac{f(x)}{x} = 0. \tag{2}$$

According to a well-known theorem of M. Cartwright, in the case where the degree of $f(z)$ is strictly less than σ, condition (1) implies the uniform boundedness of this function on the entire real axis. For the whole class $B_\sigma^{(1)}$, which is substantially broader than the class B_σ of entire functions of degree $\leqslant \sigma$ bounded on the real axis, such a statement is no longer true. At the same time it turns out that, for special values of the "amplitudes" $a_k \neq 0$ and "phases" τ_k $(k = 1, 2, \dots, n)$, for any function $f(z) \in B_\sigma^{(1)}$ the sum $\sum_{k=1}^{n} a_k f(x + \tau_k)$ always represents an entire function of degree $\leqslant \sigma$, bounded on the real axis, i.e.

$$\sum_{k=1}^{n} a_k f(x + \tau_k) \in B_\sigma. \tag{3}$$

This phenomenon in the simplest case was first discovered by S. N. Bernstein ([1]), who proved that for any function $f(z) \in B_\sigma^{(1)}$ the inequalities

$$\left| f\left(x - \frac{\pi}{2\sigma}\right) + f\left(x + \frac{\pi}{2\sigma}\right) \right| \leqslant \frac{8}{\pi} \sup_k \left| f\left(\frac{k\pi}{\sigma}\right) \right|,$$

$$\left|f\left(x-\frac{\pi}{\sigma}\right)-f\left(x+\frac{\pi}{\sigma}\right)\right|<\frac{16}{\pi}\sup_k\left|f\left(\frac{k\pi}{\sigma}\right)\right|$$

hold.

In connection with this result of S. N. Bernstein, a number of problems concerning interference arose, in particular the problem of the conditions that a function $\rho(t)$ must satisfy so that, for any function $f(z)\in B_\sigma^{(1)}$, the integral

$$F(f,z)=\int_{-\infty}^{\infty} f(z+t)\,d\rho(t) \tag{4}$$

be bounded on the real axis. This problem was solved by A. F. Timan ([3,4]), whose results were subsequently generalized by R. Boas ([2]). In the case where $\rho(t)$ is a step function with a finite number of jumps, the integral (4) turns into the sum (3), and the question reduces to what the amplitudes and phases must be for interference to occur.

The simplest type of such interference has the form

$$f(x)+f(x+\tau). \tag{5}$$

In this case interference is ensured if and only if the phase has the form $\tau=m\pi/\sigma$, where m is any odd number. For such τ, as was proved by A. F. Timan ([3]), the exact inequality

$$\frac{1}{2}\left|f\left(x-\frac{m\pi}{2\sigma}\right)+f\left(x+\frac{m\pi}{2\sigma}\right)\right|\le\frac{4M}{m\pi}+\frac{8Mm}{\pi}\sum_{\nu=1}^{\frac{m-1}{2}}\frac{1}{m^2-4\nu^2}$$

holds on the whole class $B_\sigma^{(1)}$. This inequality means that, as the phase increases, the interference on the class $B_\sigma^{(1)}$ deteriorates, and this deterioration occurs according to the asymptotic law

$$\frac{1}{2}\sup_{f(z)\in B_\sigma^{(1)}}\left|f\left(x-\frac{m\pi}{2\sigma}\right)+f\left(x+\frac{m\pi}{2\sigma}\right)\right|=\frac{2M}{\pi}\ln m+O(1). \tag{6}$$

The question arises whether, starting from the simplest case (5), one cannot qualitatively improve the interference by iteration. It turns out that the following proposition holds here.

Theorem 1. *For a given value of M and unbounded increase of m (odd), the asymptotic equality*

$$\sup_{f(z)\in B_\sigma^{(1)}} \left| \frac{1}{2^n} \sum_{\nu=0}^{n} \binom{n}{\nu} f\left(x + \nu \frac{m\pi}{\sigma}\right) \right| = \frac{2M}{\pi} \ln m + O(1)$$

holds.

Thus, comparing this result with the result of A. F. Timan (6), we see that iteration of sums (5) does not qualitatively improve the interference.

Considering the general case (3) under the condition that all amplitudes are the same and the number of phases is even, we see that interference is easy to ensure if the terms are grouped in pairs. In the case when the number of phases is odd, interference can be ensured by selecting three terms and grouping the remaining ones two by two.

Thus, alongside the simplest type of interference of the form (5), a special place is occupied by the second simplest type of interference, of the form

$$f(x-\tau) + f(x) + f(x+\tau). \tag{7}$$

Here, as in the first simplest case, first of all the question arises as to what the phase τ must be in order that interference take place on the class $B_\sigma^{(1)}$, and, secondly, according to what asymptotic law the deterioration of the interference occurs as τ increases. The answer to these questions is

Lemma. *The sum $f(x-\tau) + f(x) + f(x+\tau)$ is bounded on the real axis for every function $f(z) \in B_\sigma^{(1)}$ if and only if*

$$\tau = \frac{2(3m+1)\pi}{3\sigma} \quad \text{or} \quad \tau = \frac{2(3m+2)\pi}{3\sigma},$$

where m is any integer.

Theorem 2. *For a given value of M and unbounded increase of $m = 3p + i$ $(i = 1, 2)$, the asymptotic equality*

$$\frac{1}{3} \sup_{f(z)\in B_\sigma} \left| f\left(x - \frac{2m\pi}{3\sigma}\right) + f(x) + f\left(x + \frac{2m\pi}{3\sigma}\right) \right| = \frac{4M}{3\pi} \ln m + O(1) \tag{8}$$

holds.

Comparison of Theorem 2 with equality (6) shows that, for $m \to \infty$, in passing from $n = 2$ to $n = 3$, the interference is qualitatively improved. It is of interest to investigate the asymptotic law of change of interference with further increase in the number n of equidistant phases.

Lemma 1. *In the case of equidistant nodes, for every function $f(z) \in B_\sigma^{(1)}$ the sum $\frac{1}{n}\sum_{k=1}^{n} f(x+\tau_k)$ is bounded on the real axis if and only if*

$$\tau_k = x_0 + \frac{2m\pi}{n\sigma}k, \quad \text{where } m = np + i \quad (i = 1, 2, \dots, n-1).$$

Theorem 3. For a given value of M, even n, and unbounded increase of $m = np \pm 1$, the following asymptotic equality holds:

$$\frac{1}{n} \sup_{f(z)\in B_\sigma^{(1)}} \left| f\left(x - \frac{n-1}{n\sigma}m\pi\right) + \dots + f\left(x - \frac{m\pi}{n\sigma}\right) + \right.$$

$$\left. + f\left(x + \frac{m\pi}{n\sigma}\right) + \dots + f\left(x + \frac{n-1}{n\sigma}m\pi\right) \right| = \frac{4M}{n\pi \sin\frac{\pi}{n}} \ln m + O(1).$$

Theorem 4. For a given value of M, odd n, and unbounded increase of $m = np \pm 1$, the following asymptotic equality holds:

$$\frac{1}{n} \sup_{f(z)\in B_\sigma^{(1)}} \left| f\left(x - \frac{n-1}{n\sigma}m\pi\right) + \dots + f\left(x - \frac{2m\pi}{n\sigma}\right) + \right.$$

$$\left. + f(x) + f\left(x + \frac{2m\pi}{n\sigma}\right) + \dots + f\left(x + \frac{n-1}{n\sigma}m\pi\right) \right| = \frac{2M}{n\pi \sin\frac{\pi}{2n}} \ln m + O(1).$$

Theorems 3 and 4 show that, as the number of phases grows, in the case of even and odd n, taken separately, the interference improves qualitatively.

We shall indicate one more inequality which generalizes and sharpens the second of the estimates of S. N. Bernstein given above.

Theorem 5. For every function $f(z) \in B_\sigma^{(1)}$ and every m, the inequality

$$\frac{1}{2} \left| f\left(x - \frac{m\pi}{\sigma}\right) - f\left(x + \frac{m\pi}{\sigma}\right) \right| \le \frac{8Mm}{\pi} \sum_{\nu=-m+1}^{m} \frac{1}{4m^2 - (2\nu - 1)^2}. \quad (9)$$

holds.

For every m there exists a function $f(z) \in B\sigma$ for which the left-hand side of (9) is exactly equal to the right-hand side for $x = \pi/2\sigma$.

Corollary 1. For a given value of M and for unbounded increase of m, the asymptotic equality

$$\frac{1}{2} \sup_{f(z)\in B_\sigma^{(1)}} \left| f\left(x - \frac{m\pi}{\sigma}\right) - f\left(x + \frac{m\pi}{\sigma}\right) \right| = \frac{2M}{\pi} \ln m + O(1).$$

holds.

Corollary 2. Putting $m = 1$ in inequality (9), we obtain

$$\left| f\left(x - \frac{\pi}{\sigma}\right) - f\left(x + \frac{\pi}{\sigma}\right) \right| \leq \frac{32M}{3\pi}.$$

In conclusion, I express my sincere gratitude to Prof. A. F. Timan for posing the problem and for his guidance in its solution.

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