



Soviet-era science, translated into English

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1964

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Abstract

Full Text

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ON THE INSTABILITY OF DYNAMICAL SYSTEMS WITH MANY DEGREES OF FREEDOM

(Presented by Academician A. N. Kolmogorov on 14 II 1964)

§ 1. Recent progress in perturbation theory has made it possible to find many conditionally periodic motions in every nonlinear dynamical system close to an integrable one (see ^(1, 2)). The stability of all motions of a system follows from these results only in the cases when the dimension of the phase space is ≤ 4 . The aim of the present note is to indicate an **example (3) of a system with a 5-dimensional phase space, satisfying all the conditions of the works ^(1, 2), but unstable***. The secular changes of I_2 in the system (3) have rate $\exp(-1/\sqrt{\varepsilon})$ and therefore cannot be captured by any approximation of the classical perturbation theory.

First we introduce some definitions.

§ 2. **Whiskered torus.** By a torus T^k we mean the direct product of k circles, admitting k angular coordinates $\varphi = \varphi_1, \dots, \varphi_k \pmod{2\pi}$. A **conditionally periodic motion with frequencies** ω is defined by the equations $\dot{\varphi} = \omega = \text{const}$ (where $\sum n_i \omega_i \neq 0$ for integers n_i , $\sum n_i^2 \neq 0$). Suppose that in the phase space of a dynamical system there is an invariant torus T , and on it the motion is conditionally periodic. We shall call T a **whiskered torus** if T is a component of the intersection of two invariant open manifolds Y^- , Y^+ , and all trajectories on the **incoming whisker** Y^- tend to T as $t \rightarrow +\infty$, while on the **outgoing whisker** Y^+ they tend to T as $t \rightarrow -\infty$.

Example 1. We shall call the torus $x = y = z = 0$ in the system

$$\dot{x} = \lambda x, \quad \dot{y} = -\mu y, \quad \dot{z} = 0, \quad \dot{\varphi} = \omega, \quad (1)$$

defined in the $(l_+ + l_- + l_0 + k)$ -dimensional space x, y, z, φ ($\varphi \equiv \text{mod } 2\pi$), a **standard whiskered torus**.

For what follows the concept of a **screening set** is essential. Let M be a smooth submanifold of the space X . We shall denote the tangent plane to M at the point x by TM_x . A manifold N **complements** M at the point $x \in M \cap N$ if $TM_x + TN_x = TX_x$. We shall say that a set Ω **screens** a manifold M at the point $x \in M$, if every manifold N that complements M at x intersects Ω .

Example 2. A spiral Ω , winding onto a closed curve M , screens it**.

Another example is supplied by the standard whiskered torus (1). Let U be a neighborhood of the point ξ of the incoming whisker $x = z = 0$. Denote by

$$\Omega = \bigcup_{t>0} U(t)$$

the set of all points of all trajectories beginning in U . It is easily proved

* In contrast to stability, instability is stable. It seems to us that the mechanism of “transition chains,” ensuring instability in our example, also operates in the general case (for example, in the three-body problem).

** The works (3, 4) are based on this circumstance.

Theorem 1. *The set Ω blocks the outgoing whisker $y = z = 0$ at each of its points η .*

§ 3. **Transition chain.** If a whiskered torus T has the property that the images of any neighborhood of any point ξ of its incoming whisker block the outgoing whisker at any point of the latter, η , then we shall call such a torus **transitional**. According to Theorem 1, the **standard torus** (1) is transitional.

Suppose that a dynamical system with phase space X has several transitional tori T_1, \dots, T_s, \dots . We shall call these tori a **transition chain** if the outgoing whisker Y_s^+ of each preceding torus T_s supplements the incoming whisker of the next torus Y_{s+1}^- at some point of their intersection

$$x_s \in Y_s^+ \cap Y_{s+1}^-.$$

Let T_1, \dots, T_s, \dots be a transition chain. It is easily proved that

Theorem 2. *Any neighborhood of the torus T_1 is connected with any neighborhood of the torus T_s by a trajectory of the dynamical system under consideration.*

Thus, to prove instability it is enough to find a transition chain joining distant tori T_1, T_s . The determination of whiskered tori and, especially, the study of their intersections in the general problem of perturbation theory require cumbersome calculations. We shall restrict ourselves to an example in which a specially chosen perturbation vanishes on the tori T_s .

§ 4. **An unstable system.** We shall consider a system with two degrees of freedom, periodic in time t with period 2π . The “phase space” $I_1, I_2; \varphi_1, \varphi_2; t$ is the direct product of the plane I_1, I_2 and the three-dimensional torus $\varphi_1, \varphi_2, t \pmod{2\pi}$. The Hamiltonian function, depending on the parameters ε, μ , will have the form $H = H_0 + \varepsilon H_1$, where*

$$H_0 = \frac{1}{2} (I_1^2 + I_2^2), \quad \varepsilon H_1 = \varepsilon (\cos \varphi_1 - 1) [1 + \mu B], \quad B = \sin \varphi_2 + \cos t. \quad (2)$$

In other words, the following system of differential equations is considered:

$$\begin{aligned} \dot{\varphi}_1 = I_1, \quad \dot{\varphi}_2 = I_2; \quad \dot{I}_1 = \varepsilon \sin \varphi_1 [1 + \mu B], \quad \dot{I}_2 = \varepsilon (1 - \cos \varphi_1) \mu \cos \varphi_2; \\ B = \sin \varphi_2 + \cos t. \end{aligned} \quad (3)$$

We first study the unperturbed system ($\varepsilon = 0$). Each three-dimensional torus $I_1 = \omega_1$, $I_2 = \omega_2$ is invariant. On it there occurs a three-frequency motion

$$\dot{\varphi}_1 = \omega_1, \quad \dot{\varphi}_2 = \omega_2, \quad \dot{t} = 1.$$

The torus is called **nonresonant** if the frequencies on it are independent (i.e. $n_1\omega_1 + n_2\omega_2 + n_0 \neq 0$ for integral $n \neq 0$). The equation $I_1 = 0$ determines a family of resonant tori (since $\omega_1 = 0$).

Now consider the perturbed system: let $0 < \varepsilon\mu \ll \varepsilon \ll 1$. In ^(1,2) it is shown that, for the majority of nonresonant initial conditions, the quantities $I_1(t), I_2(t)$ change little throughout the entire infinite interval of time $-\infty < t < +\infty$. It turns out, however, that near the resonant manifold $I_1 = 0$ there appears a zone of instability. More precisely, the following holds.

Theorem 3. *Let $0 < A < B$. For every $\varepsilon > 0$ there exists $\mu_0 > 0$ such that, for $0 < \mu < \mu_0$, system (3) is unstable: there exists a trajectory joining the region $I_2 < A$ with the region $I_2 > B$.*

§ 5. **Proof of instability.** Fix $\varepsilon > 0$.

A. **First let $\mu = 0$.** Then the variables separate:

$$H = H^{(1)} + H^{(2)}, \quad H^{(1)} = \frac{1}{2}I_1^2 + \varepsilon(\cos \varphi_1 - 1), \quad H^{(2)} = \frac{1}{2}I_2^2. \quad (4)$$

Thus $\dot{I}_2 = 0$, $\dot{\varphi}_2 = I_2 = \omega = \text{const}$, while the change of I_1, φ_1 with time is described by the Hamiltonian of an ordinary pendulum $H^{(1)}$. Let the number ω be irrational. It is easily proved

* It is not difficult to construct a real mechanical system with Hamiltonian function (2).

Assertion A. The manifold T_ω , defined by the equations $I_1 = \varphi_1 = I_2 - \omega = 0$, is a two-dimensional hyperbolic torus of system (3). The whiskers are three-dimensional and have equations

$$H^{(1)} = 0, \quad H^{(2)} = \frac{1}{2}\omega^2 \quad \text{or} \quad I_1 = \pm 2\sqrt{\varepsilon} \sin \frac{\varphi_1}{2}, \quad I_2 = \omega. \quad (5)$$

The whiskers are filled with asymptotic trajectories

$$I_1(t) = \pm 2\sqrt{\varepsilon} \operatorname{ch}^{-1} \tau, \quad \varphi_1(t) = \pm \operatorname{arcctg}(-\operatorname{sh} \tau), \quad \varphi_2(t) = \varphi_2^0 + \omega(t - t^0), \quad (6)$$

where $\tau = \sqrt{\varepsilon}(t - t^0)$, $I_1(t^0) = \pm 2\sqrt{\varepsilon}$, $\varphi_1(t^0) = \pm\pi$, $\varphi_2(t^0) = \varphi_2^0$.

Thus, a point of the outgoing whisker of the torus T_ω returns, as $t \rightarrow +\infty$, to the same torus T_ω . In other words, the outgoing whisker forms a single manifold with the incoming one. Naturally, for $\mu \neq 0$ this manifold splits into two whiskers intersecting each other.* We shall see that (in contrast to separatrices of systems with phase space of dimension ≤ 4 , considered in (5, 6)) these whiskers also intersect the whiskers of neighboring tori T_ω .

B. Let now $\mu \neq 0$. From (3) it is clear that the tori T_ω remain invariant for all μ . Let ω be irrational. By the standard method of contracted mappings one proves**

Assertion B. The manifold T_ω is a hyperbolic transition torus of system (3), if μ is sufficiently small.

Let $\omega_1 < A < B < \omega_s$. To prove Theorem 3 it is sufficient to construct a transition chain of tori $T_{\omega_1}, \dots, T_{\omega_s}$ and use Theorem 2. The construction of such a chain is based on the investigation of the perturbation of the whiskers (5) for small μ . It turns out that the following is true.

Lemma 1. Let $A < \omega < B$. Then the outgoing whisker Y_ω^+ of the torus T_ω intersects the incoming whiskers $Y_{\omega'}^-$ of all tori $T_{\omega'}$ so close that $|\omega - \omega'| \leq \varkappa$ (where $\varkappa = \varkappa(\varepsilon, \mu, A, B) > 0$).

The proof of Lemma 1 requires some calculations. The unperturbed whiskers have equations (5): $H^{(1)} = 0$, $H^{(2)} = \frac{1}{2}\omega^2$, where $H^{(k)}$ are the functions (4). Let $\alpha > 0$ (for example, $\alpha = \pi/2$). It is easy to show that, for $|\varphi_1| < 2\pi - \alpha$, the equations of the perturbed outgoing whisker Y_ω^+ can be written in the form

$$H^{(1)} = \Delta_1^+(\varphi_1; \varphi_2, t; \omega); \quad H^{(2)} = \frac{1}{2}\omega^2 + \Delta_2^+(\varphi_1; \varphi_2, t; \omega), \quad (7)$$

where the functions $\Delta_k^+ = O(\mu)$ have period 2π in φ_2, t and are equal to 0 when $\varphi_1 = 0$. In exactly the same way, the incoming whisker $Y_{\omega'}^-$, for $|\varphi_1 - 2\pi| < 2\pi - \alpha$, has equations

$$H^{(1)} = \Delta_1^-(\varphi_1; \varphi_2, t; \omega'), \quad H^{(2)} = \frac{1}{2}\omega'^2 + \Delta_2^-(\varphi_1; \varphi_2, t; \omega'). \quad (8)$$

We shall seek the intersection of the whiskers Y_ω^+ and $Y_{\omega'}^-$ in the plane $\varphi_1 = \pi$. In the notation (7), (8), Lemma 1 is the assertion of solvability, with respect to φ_2, t , of the system of equations

$$\Delta_1^+(\pi; \varphi_2, t; \omega) = \Delta_1^-(\pi; \varphi_2, t; \omega'), \quad (9)$$

$$\frac{1}{2}\omega^2 + \Delta_2^+(\pi; \varphi_2, t; \omega) = \Delta_2^-(\pi; \varphi_2, t; \omega') + \frac{1}{2}\omega'^2.$$

The solvability of system (9) is derived from the following approximate expressions for Δ_k^\pm .

* The splitting of separatrices was studied by Poincaré in the last chapter of *New Methods* (5). Poincaré's investigations have recently been continued by Melnikov (6).

** A convenient conical metric is $\|f(x)\| = \max |x^{-1}f(x)|$.

Lemma 2 (cf. (6)). *The perturbations of the whiskers are*

$$\Delta_k^\pm = \mu\delta_k^\pm + O(\mu^2),$$

where

$$\mu\delta_k^\pm(\pi; \varphi_2^0, t^0; \omega) = \int_{\mp\infty}^0 \{H, H^{(k)}\} d(t-t^0) \Big|_{(6)} \quad (10)$$

(the Poisson bracket is integrated along the unperturbed trajectory (6)).

Indeed, according to the definitions (7), (8), the quantities Δ_k^\pm are the increments of $H^{(k)}$ in the perturbed motion (3). The derivative of the function $H^{(k)}$ by virtue of the system of equations (3) is precisely the Poisson bracket $\{H, H^{(k)}\}$. Therefore Δ_k^\pm are exactly equal to the integrals (10), extended to the perturbed trajectories. Hence one readily obtains the estimate $\Delta_k^\pm - \mu\delta_k^\pm = O(\mu^2)$, which proves Lemma 2.

It follows from Lemma 2 that the solvability of the system (9) depends mainly on the solvability, with respect to φ_2^0, t^0 , of the approximate system

$$\delta_1 = 0, \quad \mu\delta_2 = \frac{1}{2}(\omega^2 - \omega a'^2), \quad (11)$$

where

$$\delta_k = \delta_k^+(\pi; \varphi_2^0, t^0, \omega) - \delta_k^-(\pi; \varphi_2^0, t^0, \omega) = \int_{-\infty}^{+\infty} \{H, H^{(k)}\} d(t-t^0) \Big|_{(6)}. \quad (12)$$

Straightforward computations based on formulas (2)–(6) give

$$\delta_1 = -2\varepsilon \int_{-\infty}^{+\infty} u \frac{\partial B}{\partial t} dt, \quad \delta_2 = 2\varepsilon\omega \int_{-\infty}^{+\infty} u \frac{\partial B}{\partial \varphi_2} dt, \quad (13)$$

where $u = \text{ch}^{-2} \tau$, $\tau = \sqrt{\varepsilon}(t-t^0)$, $B = B(\varphi_2, t)$, $\varphi_2 = \varphi_2^0 + \omega(t-t^0)$. For $B = \sin \varphi_2 + \cos t$ the integrals (13) are evaluated by residues*:

$$\delta_1 = 2\pi \left(\operatorname{sh}^{-1} \frac{\pi}{2\sqrt{\varepsilon}} \right) \sin t^0, \quad \delta_2 = 2\pi\omega^2 \left(\operatorname{sh}^{-1} \frac{\omega\pi}{2\sqrt{\varepsilon}} \right) \cos \varphi_2^0. \quad (14)$$

Putting $t^0 = 0$ in (14), we verify the solvability of the system (11) when

$$|\omega^2 - \omega a'^2| < 4\pi\mu\omega^2 \operatorname{sh}^{-1} \frac{\omega\pi}{2\sqrt{\varepsilon}} \sim \mu e^{-1/\sqrt{\varepsilon}}. \quad (15)$$

It now follows from Lemma 2 that, for sufficiently small μ , the system (9) is also solvable. From the inequality (15) one easily obtains the uniform, for $A < \omega < B$, estimate from below of $\max_{\omega'} |\omega - \omega'| = \chi(\omega)$ required in Lemma 1.

Thus Lemma 1 is proved. It allows one to construct a chain of transition tori $T_{\omega_1}, \dots, T_{\omega_s}$ ($\omega_1 < A < B < \omega_s$). From formulas (14) it is clear that, for sufficiently small μ , this chain can be chosen so that consecutive intersecting whiskers lie in general position and complement one another in the sense of § 2. Then the chain $T_{\omega_1}, \dots, T_{\omega_s}$ will be transitional. The application of Theorem 2 to the transitional chain $T_{\omega_1}, \dots, T_{\omega_s}$ completes the proof of Theorem 3.

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Received
30 I 1964

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* The analogous integrals in (6), p. 32, were computed incorrectly.

Note: Figure translations are in progress. See original paper for figures.

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