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Abstract

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ON TRANSITION THROUGH THE SPEED OF SOUND IN LAVAL NOZZLES WITH A CIRCULAR CROSS SECTION

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The equations describing an axisymmetric transonic flow,

$$-u\,\partial u/\partial x + \partial v/\partial r + v/r = 0, \qquad \partial u/\partial r = \partial v/\partial x \tag{1}$$

were obtained by T. von Kármán (^{1}). We shall apply them to the study of the features of transition through the speed of sound in Laval nozzles with a circular cross section. For this purpose we prescribe, along the nozzle axis, i.e., for $r = 0$, the distribution of the components of the particle velocity in the form

$$u = -A_1|x|^k \quad \text{for } x < 0; \qquad u = A_2 x^k \quad \text{for } x > 0, \qquad v = 0$$
$$(A_1 > 0,\ A_2 > 0). \tag{2}$$

We shall assume that the values of the exponent k lie in the interval $1 < k < 2$; in the corresponding gas motions the sonic line is a power curve concave toward the oncoming flow. Problem (2) with $k = 1$ was studied in detail in papers (^{2}, ^{3}). For $k = 2$ the transition line becomes a straight line perpendicular to the axis of symmetry of the nozzle (^{4}-^{6}).

If shock waves arise in the flows, then the solutions of the system of equations (1) must satisfy, in addition to the initial data (2), additional boundary conditions on the wave front: the equation of the shock polar (^{7})

$$2(v_2 - v_3)^2 = (u_2 - u_3)^2(u_2 + u_3) \tag{3}$$

and the relation (^{5})

$$u_2\,dx_2/dr + v_2 = u_3\,dx_2/dr + v_3, \tag{4}$$

equivalent to the condition of continuity of the tangential component of the velocity vector. In equalities (3) and (4) the indices refer to the parameters on different sides of the shock front, while $x_2 = x_2(r)$ is the equation specifying its position.

It is easy to show that the desired solution of the Cauchy problem (2) is self-similar:

$$u = r^{2(n-1)} f(\xi), \qquad v = r^{3(n-1)} g(\xi), \qquad \xi = x/r^n, \qquad n = 2/(2-k),$$

and the equation of the shock front has the form $\xi = \xi_2 = \mathrm{const}$.

Substitution of the written formulas into the system of equations (1) and elimination of the function $g(\xi)$ from the relations obtained give, for determining $f(\xi)$, the second-order differential equation

$$(f - n^2\xi^2)\, d^2f/d\xi^2 + (df/d\xi)^2 + n(3n-4)\xi\, df/d\xi - 4(n-1)^2 f = 0. \tag{5}$$

To simplify the qualitative investigation of the problem under consideration, we set ([2], [3])

$$f = \xi^2 F(\eta), \qquad dF/d\eta = \Psi, \qquad \eta = \ln|\xi|. \tag{6}$$

In the new variables the order of equation (5) is reduced:

$$d\Psi/dF = \frac{-4F - 4n\Psi - 6F^2 + 7F\Psi + \Psi^2}{(n^2 - F)\Psi}. \tag{7}$$

The Cauchy data (2) lead to the requirement that the integral curve of equation (7), representing the velocity field in a neighborhood of the nozzle throat, begin and end at its singular point $A(0,0)$, which corresponds to the x-axis. In a neighborhood of A this curve is given by the expansion

$$\Psi = -\frac{2}{n}F - 3\frac{(n-\frac{4}{3})(n-1)}{n^3}F^2 + \frac{3}{4}\frac{(n-\frac{4}{3})(n-1)(6n^2-7n-2)}{n^5}F^2 + \ldots \tag{8}$$

The boundary conditions (3) and (4) in the plane $F\Psi$ are written as

$$F_2 + F_3 = 2n^2, \qquad \Psi_2 + \Psi_3 = -2n(7n-4). \tag{9}$$

Motion along an integral curve in the plane $F\Psi$ from the point A in the direction of the node $C\left[n^2,\ n\left(4 - 7n + \sqrt{25n^2 - 56n + 32}\right)/2\right]$ represents the gas flow in the inlet part of the nozzle between the x-axis and the C_-^0-characteristic

Fig. 1

Figure 1: Fig. 1

arriving at its center. Passage through the point C means intersection of the C_-^0-characteristic in the physical plane. By successive differentiation of formulas (6) it is easy to show that a discontinuity in the i-th derivative of the function $\Psi(F)$ corresponds to discontinuities in $(i+1)$ derivatives of the velocity-vector components with respect to the coordinates. Therefore the character of the flow singularity on the C_-^0-characteristic is determined by the expansion of the function $\Psi(F)$ in a neighborhood of the point C

$$\Psi = \frac{1}{2} n \left(4 - 7n + \sqrt{25n^2 - 56n + 32}\right) +$$

$$+ a_1(F - n^2) + a_2(F - n^2)^2 + \ldots + b_1(F - n^2)^\lambda + \ldots \tag{10}$$

Here the coefficients a_i depend only on n, the constant b_1 is arbitrary, and the exponent λ of the first term of the irregular part is given by the formula

$$\lambda = -2\sqrt{25n^2 - 56n + 32} \Big/ \left(4 - 7n + \sqrt{25n^2 - 56n + 32}\right). \tag{11}$$

As is known, the exponent λ is equal to the ratio of the roots of the characteristic equation determining the type of the singular point C. So long as its value is not an integer positive number, then, according to the theorem of Briot and Bouquet (^{8}), only one of the integrals (10) is holomorphic. It is obtained by equating the arbitrary constant b_1 to zero. On the contrary, for integral values of λ, either all integrals of the ordinary differential equation are representable by a Taylor series, or none of them is holomorphic. In the latter case representation (10) loses its validity; logarithmic terms appear in it. Analysis of equation (7) shows that, for positive integral λ, in a neighborhood of the point C there is not a single holomorphic integral, i.e., the second possibility admitted by the theorem of Briot and Bouquet is realized.

Fig. 1

Let us construct, by means of integrals corresponding to nonintegral values of the exponent λ, flows without singularities in the derivatives of the velocity components with respect to the coordinates on the C_-^0-characteristic. In them the streamlines at the points of intersection with the C_-^0-characteristic also have no singularities; therefore, in order to realize flows analytic in a neighborhood of the C_-^0-characteristic, there is no need to resort to any artificial devices by giving the nozzle walls a special shape. Calculations carried out on the high-speed

Fig. 2

Figure 2: Fig. 2

Fig. 3

Figure 3: Fig. 3

electronic computer BESM-2 show that construction of a flow field without singularities on the characteristic closing the inlet part of the channel is indeed possible for $k = 1.167$, $n = 2.366$, and $A_2 = 0.629A_1$. Along the C^0_+-characteristic issuing from the center in such a flow, discontinuities of the third derivatives of the components of the velocity vector propagate. Axisymmetric flows with shock waves arising at the center in nozzles with smooth walls do not exist.

The flow realized for $k = 1.157$ and $A_2 = 0.629A_1$, like the analytic flow with $k = 1$ and $A_2 = A_1$ studied in papers ([2],[3]), has an asymptotic character in a neighborhood of the nozzle center. The singularities present in it originate not at the walls but in the flow itself, at the point of interse-

of the sound line with the axis of symmetry, and are then carried into the exhaust part of the channel.

In the two indicated types of asymptotic flows, the velocity fields are substantially different. Along each streamline in the flow with $k = 1$ the velocity increases monotonically. Conversely, in the gas motion with $k = 1.157$, the distribution of velocity along the streamlines has two relative extrema: a maximum between the sound line and the C^0_--characteristic and a minimum in the region enclosed between the C^0_+-characteristics. In connection with this, the form of the curves $u = \text{const}$ for $k = 1.157$ will have the form shown in Fig. 1.

Fig. 2

In all the remaining continuous as well as discontinuous flows, along the C^0_--characteristic closing the inlet part of the nozzle, one or another singularity occurs in the derivatives of the velocity components with respect to the coordinates. In formula (11), the quantity $\lambda = 1, 2, 3, 4$, respectively, for $k = 12/17$, $(7+2\sqrt{21})/14$, $(26+25\sqrt{2})/41$, $3(7+\sqrt{91})/28$, and $n = 17/11$, $4(21+2\sqrt{21})/51$, $(56+25\sqrt{2})/23$, $4(35+3\sqrt{91})/29$. When the values $k \to 2$ and $n \to \infty$, the exponent $\lambda \to 5$. Hence we conclude that on the C^0_--characteristic, for $1 \leq k \leq (7+2\sqrt{21})/14$, all derivatives of the velocity components, beginning with the third, have infinite jumps; when $(7+2\sqrt{21})/14 < k \leq (26+25\sqrt{2})/41$, infinite jumps arise in the fourth and higher derivatives; for $(26+25\sqrt{2})/41 < k \leq 3(7+\sqrt{91})/28$, the fifth and higher derivatives have infinite jumps; finally, when $3(7+\sqrt{91})/28 < k < 2$, all derivatives of the components of the velocity vector, beginning with the sixth, undergo infinite jumps. For $k = (7+2\sqrt{21})/14$, $(26+25\sqrt{2})/41$, $3(7+\sqrt{91})/28$, expansion (10) loses its validity; it must be replaced by an expansion containing logarithmic terms.

Fig. 3

To obtain the flow expanding most rapidly beyond the C_+^0-characteristic, in the phase plane $F\Psi$ it is necessary to make a jump from the point C to the saddle

$$D\left[n^2,\ n\left(4-7n-\sqrt{25n^2-56n+32}\right)/2\right],$$

then move along the separatrix passing through it in the direction of the infinitely distant point E, which is located on the straight line $\Psi = -2F$, and again return along the continuation of the separatrix to the point C. Motion along the indicated integral curve of equation (7) in the reverse direction, from the point C through E to the point D, represents the flow which, among all continuous ones, expands most slowly beyond the C_+^0-characteristic. The construction of limiting flows makes it possible to find the range of variation of the ratio A_2/A_1 corresponding to shock-free motions of the gas. The boundaries of this region are shown in Fig. 2. In the flow with maximum expansion in the direction of the nozzle exhaust, jumps of accelerations are formed on the C_-^0-characteristic; in the most slowly expanding flow, on the C_+^0-characteristic.

In discontinuous flows, the compression shock originates at the center of the channel and is then carried downstream. To construct discontinuous flows it is necessary to choose integral curves of equation (7) that emerge from the point C in the direction of the point E, and whose continuations, beginning at E, are located below the separatrix passing through the saddle D. Along such

along the curves as $F \to n^2$ the values $\Psi \to -\infty$, and in the corresponding gas motions limiting lines arise that are the envelopes of the C_+^0-characteristics and carry infinite values of the accelerations. Since a flow with infinite accelerations is physically meaningless, a shock wave must form in it before the limiting line appears. But it proves impossible to introduce a shock wave into a flow in which there are no infinite accelerations. Let us note that, when a compression shock arises, the gas motion in the inlet part of the nozzle is not disturbed.

The gas flow behind the compression shock, in the variables F, Ψ, must be represented by a segment of the integral curve (8). This condition, together with equalities (9), determines the intensity of the shock wave. For computations it is simplest to use direct integration of the original equation (5), subsequently transferring the results to the $F\Psi$-plane. As an example, for $k = 1.157$, Fig. 3 shows the dependence $f_3 - f_2$ on the constant A_2. Figure 4 gives the dependence of the coordinate ξ_2 of the compression shock on the same constant. The quantity A_1 is chosen so that the position of the C_-^0-characteristic is determined by the equality $\xi_1 = -1$. In order to cause the appearance of a shock wave directly at the point of intersection of the sonic curve with the axis of a circular nozzle, it is necessary to send disturbances there along the C_-^0-characteristic. It is true that the indicated disturbances may be very weak. Thus gas motions with

$$k = (7+2\sqrt{21})/14, \quad (26+25\sqrt{2})/41, \quad 3(7+\sqrt{91})/28$$

give examples of the formation of shock waves as a result of the reflection, from the center of the nozzle, of logarithmic singularities respectively in the third, fourth, and fifth derivatives of the velocity components with respect to the coordinates. It must be emphasized that the singularities brought along the C^0_--characteristic to the center are not of essential significance for analyzing the causes of shock-wave formation in the neighborhood of the channel constriction. Shock waves arise as a result of the appearance of an envelope of characteristics issuing from points of the sonic line and inclined downstream. If the velocity field, upon crossing a C^0_--characteristic, retains its analytic character but as a whole corresponds to a nozzle in which a continuous motion cannot be realized, then the compression shock is initiated somewhat to the right of the transition line in the supersonic region.

Fig. 4

It follows from Fig. 2 that shock waves in flows form only when the values of the ratio A_2/A_1 become lower than a certain limit. From this it is easy to conclude that, structurally, the formation of shock waves near the critical section of the nozzle is associated with an excessively long transition part. In designing nozzles the transition part should be made as short as possible; increasing the distance between the throat and the inlet of the channel leads to a slower expansion of the flow and, ultimately, to the appearance of discontinuities. In the limiting case the velocity behind the compression shock is equal in magnitude to the critical velocity and is directed along the axis of symmetry of the nozzle.

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