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# Reports of the Academy of Sciences of the USSR

MATHEMATICS

1964

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## Abstract

## Full Text

Reports of the Academy of Sciences of the USSR
1964. Volume 159, No. 4

*MATHEMATICS*

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# ON THE COMPLETENESS OF A CERTAIN PART OF THE EIGENFUNCTIONS OF A NON-SELF-ADJOINT DIFFERENTIAL OPERATOR

*(Presented by Academician I. G. Petrovskii on 29 V 1964)*

Several fundamental works ($^{1-3}$) have been devoted to the question of completeness and expansion in eigenfunctions and associated functions of non-self-adjoint differential operators with coefficients depending on a complex parameter.

In the work ($^3$), M. V. Keldysh, along with other facts, proved the $n$-fold completeness of all eigenfunctions and associated functions of an ordinary differential equation with coefficients depending polynomially on the parameter.

In the investigation of certain problems of mathematical physics and mechanics, it is necessary to approximate the boundary value of the solution by a certain part of the eigenfunctions of the corresponding non-self-adjoint differential operator.

The purpose of the present note is to show the completeness of one half (the meaning of the word half is clarified in the text) of the eigenfunctions of the problem

$$y''(x) + 2B\lambda y'(x) + C\lambda^2 y(x) = 0, \tag{1}$$

$$y'(0) + a\lambda y(0) = 0, \tag{2}$$

$$y'(1) + a\lambda y(1) = 0, \tag{3}$$

where $B, C, a$ are constant numbers, with $B^2 - C < 0$, and $\lambda$ is a parameter.

Obviously, the eigenvalues $\{\lambda_k\}$ of problem (1), (2), (3) coincide with the roots of the equation

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$$e^{i\lambda\sqrt{C-B^2}} - e^{-i\lambda\sqrt{C-B^2}} = 0.$$

Consequently,

$$\lambda_k = \frac{\pi}{\sqrt{C-B^2}}\,k = \gamma k, \qquad k = 0,\ \pm 1,\ \pm 2, \dots, \quad \gamma > 0.$$

Then the corresponding eigenfunctions have the form

$$y_k(x) = ce^{\alpha k x} + e^{\beta k x}, \qquad k = 0,\ \pm 1,\ \pm 2, \dots,$$

where

$$c = \frac{B - a + i\sqrt{C-B^2}}{-B + a + i\sqrt{C-B^2}},$$

$$\alpha = \pi\left(\frac{-B}{\sqrt{C-B^2}} + i\right), \qquad \beta = -\pi\left(\frac{B}{\sqrt{C-B^2}} + i\right).$$

Denote by $\lambda_k^+$ those eigenvalues of problem (1), (2), (3) which correspond to positive values of $\lambda_k$, and by $\lambda_k^-$ those which correspond to nonpositive values of $\lambda_k$. Let $y_k^+(x)$ be those eigenfunctions which correspond to the eigenvalues $\lambda_k^+$, and $y_k^-(x)$ those,

which correspond to $\lambda_k^-$. Then

$$y_k(x) = \begin{cases} y_k^+(x) = ce^{\alpha k x} + e^{\beta k x}, & k = 1, 2, \dots, \\ y_k^-(x) = ce^{-\alpha k x} + e^{-\beta k x}, & k = 0, 1, 2, \dots. \end{cases}$$

**Theorem.** The system of eigenfunctions $\{y_k^-(x)\}$ is complete in the space $L_2(0,1)$.

**Proof.** Let $f(x)$ be an arbitrary function from $L_2(0,1)$, and suppose

$$\int_0^1 f(x) y_k^-(x)\,dx = \int_0^1 f(x)(ce^{-\alpha k x} + e^{-\beta k x})\,dx = 0, \qquad k = 0, 1, 2, \dots.$$

Obviously, to prove the theorem it suffices to show that $f(x) = 0$. For this purpose consider the function

$$F(z) = \int_0^1 f(x)(ce^{ze^{-\alpha x}} + e^{ze^{-\beta x}})\,dx.$$

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Obviously, $F(z)$ is an analytic function, and

$$F^{(k)}(z)\big|_{z=0} = \int_0^1 f(x) y_k^-(x)\, dx = 0, \qquad k = 0, 1, 2, \dots;$$

therefore, $F(z) \equiv 0$.

Let $s$ be a complex number such that $\operatorname{Re} s > N$, where $N$ is a sufficiently large number. Consider the integral

$$\int_0^\infty e^{-zs} F(z)\, dz = 0$$

or

$$\int_0^\infty e^{-zs} \left\{ \int_0^1 f(x)(ce^{ze^{-\alpha x}} + e^{ze^{-\beta x}})\, dx \right\} dz = 0.$$

Changing the order of integration, we write the last integral as

$$\int_0^1 f(x) \left\{ \int_0^\infty \left[ ce^{-z(s-e^{-\alpha x})} + e^{-z(s-e^{-\beta x})} \right] dz \right\} dx = 0.$$

Expanding the inner integral, we obtain

$$\int_0^1 f(x) \left( \frac{c}{e^{-\alpha x} - s} + \frac{1}{e^{-\beta x} - s} \right) dx \equiv 0$$

or

$$\int_0^1 f(x) \left( \frac{ce^{\alpha x}}{1 - se^{\alpha x}} + \frac{e^{\beta x}}{1 - se^{\beta x}} \right) dx \equiv 0.$$

Since

$$\frac{1}{1 - se^{\alpha x}} = \sum_{k=0}^\infty e^{\alpha k x} s^k, \qquad \frac{1}{1 - se^{\beta x}} = \sum_{k=0}^\infty e^{\beta k x} s^k,$$

then

$$\sum_{k=1}^\infty \left\{ \int_0^1 f(x) \left( ce^{\alpha k x} + e^{\beta k x} \right) dx \right\} s^{k-1} \equiv 0, \qquad k = 1, 2, \dots$$

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Consequently,

$$\int_0^1 f(x) y_k^+(x)\, dx = 0, \qquad k = 1, 2, \ldots$$

Thus, from the condition that $f(x)$ is orthogonal to $\{y_k^-(x)\}$, it follows that it is also orthogonal to $\{y_k^+(x)\}$. Since the whole system $\{y_k(x)\}$ is complete in the space $L_2(0,1)$, it follows that $f(x) \equiv 0$.

In an analogous way one proves that, if zero is adjoined to the positive eigenvalues, then the corresponding eigenfunctions are complete in $L_2(0,1)$.

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Received
12 V 1964

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*Note: Figure translations are in progress. See original paper for figures.*

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