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Abstract

Full Text

MATHEMATICS

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THE OBLIQUE DERIVATIVE PROBLEM WITH POLYNOMIAL COEFFICIENTS

In the Euclidean n-dimensional space of variables $x_1, x_2, \dots, x_n$, consider a domain D with boundary S. We seek a function $U(x)$, $x \equiv (x_1, x_2, \dots, x_n)$, regular and harmonic in the domain D, continuous together with its first-order partial derivatives in the closed domain $\overline{D}$, and satisfying the boundary condition

$$P(y)\,\mathrm{grad}\,U(y) = f(y), \qquad y \equiv (y_1, y_2, \dots, y_n) \in S, \tag{1}$$

where $f(y)$ is a given continuous function, $P \equiv (p_1, p_2, \dots, p_n)$ is a given polynomial vector in the variables $y_1, y_2, \dots, y_n$, and by $\mathrm{grad}\,U(y)$ is meant $\lim_{x\to y} \mathrm{grad}\,U(x)$, $x \in D$ (the **oblique derivative problem**).

We again denote by $P(x)$ the polynomial continuation of $P(y)$ from the boundary S into the domain D. Let m be the degree of the polynomial $P(x)$, $x \in \overline{D}$.

Under unconditional or conditional solvability of problem (1), the expression $P(x)\,\mathrm{grad}\,U(x) \equiv V(x)$ is a regular solution in the domain D of the polyharmonic equation

$$\Delta^{m+1} V = 0, \tag{2}$$

satisfying the boundary condition

$$V(y) = f(y), \qquad y \in S. \tag{3}$$

The general representation of solutions of equation (2) regular in the domain D is given by the well-known Almansi formula (^{1,2})

$$V(x) = \sum_{k=0}^{m} |x|^{2k} V_k(x), \tag{4}$$

where $|x|^2 = x_1^2 + x_2^2 + \dots + x_n^2$, and $V_k(x)$ are arbitrary functions harmonic in the domain D.

Thus, problem (1) is reduced to finding regular harmonic solutions in the domain D of the linear first-order partial differential equation

$$P(x)\operatorname{grad} U(x) = V(x), \tag{5}$$

whose right-hand side is an arbitrary function polyharmonic in the domain D, satisfying the boundary condition (3).

It is well known that the problem of finding solutions of equation (5) and the problem of integrating the system of ordinary differential equations

$$dx - dtP(x) = 0, \tag{6}$$

where t is a scalar parameter, are equivalent problems.

If independent holomorphic first integrals $\xi_k(x)$, $k = 1, 2, \dots, n-1$, are known in the domain D, and a holomorphic particular solution $U_0(x)$ of the nonhomogeneous equation (5), then the holomorphic general solution of equation (5)

is written in the form

$$U = \Phi(\xi) + U_0(x), \tag{7}$$

where Φ is an arbitrary holomorphic function of $\xi = (\xi_1, \xi_2, \dots, \xi_{n-1})$.

The questions of the existence of $n-1$ independent holomorphic first integrals of system (6) and of a holomorphic particular solution of the nonhomogeneous equation (5) are studied in the classical Poincaré theory of system (6) (^{3}). Here it is assumed that $P(y) \neq 0$ everywhere on the boundary S of the domain D. In what follows we shall suppose that this assumption holds.

When S is the sphere $|y| = 1$, equation (5) takes the form

$$P(x)\operatorname{grad} U(x) = \sum_{k=1}^{m} (|x|^{2k} - 1)V_k(x) + V_0(x), \tag{8}$$

where $V_k(x)$, $k = 1, 2, \dots, m$, are arbitrary functions harmonic in the ball $|x| < 1$, and $V_0(x)$ is a function harmonic in the same ball and satisfying the boundary condition

$$V_0(y) = f(y), \qquad y \in S.$$

First consider the case when $P(x) \neq 0$ everywhere in the closed ball $|x| \leqslant 1$. In this case the first integrals $\xi_k(x)$, $k = 1, 2, \dots, n-1$, of system (6), holomorphic inside the ball, and the holomorphic particular solution $U_0(x)$ of equation (8) exist, and consequently the general holomorphic solution of this equation can be written in the form (7).

In order that the holomorphic function $U(x)$ defined by formula (7) be harmonic inside the ball $|x| < 1$, the function $\Phi(\xi)$ must satisfy a linear second-order differential equation of elliptic type

$$\sum_{i,k=1}^{n-1} a_{ik}(\xi)\Phi_{\xi_i\xi_k} + \sum_{i=1}^{n-1} b_i(\xi)\Phi_{\xi_i} = \omega(\xi), \tag{9}$$

where a_{ik}, b_i, and ω are completely determined functions, holomorphic in their arguments, which are expressed solely in terms of $P(x)$ and $U_0(x)$; moreover the functions $V_k(x)$, $k = 1, 2, \dots, m$, are determined uniquely from the requirement that $U(x)$ be harmonic.

Consequently, in the case under consideration problem (1) is always solvable, and the degree of its indeterminacy is characterized by the general holomorphic solution $\Phi(\xi)$ of equation (9). Hence, in turn, we conclude that for uniqueness of the solution of problem (1) it is necessary to prescribe the boundary values $U(y)$ on a certain manifold S_{n-2} of dimension $n-2$, lying on the sphere S. This conclusion remains valid also when the domain D is homeomorphic to a ball and its boundary S has a continuously varying tangent hyperplane.

It should be noted that in the case under consideration the Kronecker index, characterizing the rotation of the vector field $P(y)$, $y \in S$, is equal to zero.

Let now $x = 0$ be the unique singular point of system (6), i.e., the unique point of the ball $|x| < 1$ at which the vector $P(x)$ vanishes. This case necessarily arises if the Kronecker index of the vector field $P(y)$ is different from zero.

As is known, the singular points of system (6) are classified according to the character of the roots of the equation

$$\det(A - E\lambda) = 0,$$

where $A(x) \equiv \| dP_i/dx_k \|$, and E is the identity matrix.

For $n = 2$ we shall be dealing with three types of singular points: saddle, node, and focus, with the Kronecker index χ of the vector field $P(y)$ in the first case being negative, and in the other two cases positive.

It is well known that in the case of a saddle problem (1) is always solvable and the number of its linearly independent solutions is equal to $-2\chi + 2$. In the cases of a node or ⋯

of the focus, problem (1) is solvable only when $2\chi - 1$ integral conditions are imposed on f.

As n increases, the number of types of singular points grows quite rapidly. For example, for $n = 3$, alongside the saddle, node, and focus, a new type of singular point appears: the saddle-focus.

It is not difficult to show that, in the case of the sphere $|y| = 1$, only at a singular point of saddle type is there unconditional solvability of problem (1); moreover, the degree of indeterminacy of this problem, as compared with the case $P(x) \neq 0,\ x \in D$, increases on account of the arbitrariness that arises from the nonuniqueness of the determination of the functions $V_k(x),\ k = 1, 2, \dots, m$, in the right-hand side of (8). In the cases of a node and a focus, unconditional solvability of problem (1) does not occur. Unconditional solvability of this problem is, in general, also absent in the case of a saddle-focus; moreover, this time the degree of indeterminacy is greater than in the cases of a node and a focus. In all the last three cases the number of integral solvability conditions for problem (1), imposed on the function $f(y)$, is finite. These assertions remain valid for a simply connected domain D bounded by a Lyapunov surface, provided that the right-hand side f of condition (1) is required to be Hölder continuous.

Examples illustrating the case $P(x) \neq 0,\ x \in D$, and the case of a singular point of node type, were considered in our note (4). The case of a singular point of saddle type can be illustrated by the example in which the components of the vector $P(y)$ are $y_1, y_2, -y_3$. This time, for unique solvability of problem (1) in the case of the sphere $|y| = 1$, one must additionally prescribe the boundary values of the function U on the circles $y_3 = \pm 1/\sqrt{2}$, $y_1^2 + y_2^2 + y_3^2 = 1$, along which the vector $P(y)$ lies in the tangent plane S. In the example under consideration, as is easy to see, the Kronecker index of the vector field $P(y)$ is equal to minus one.

The problem studied in the present note makes it possible to construct a theory for one class of multidimensional singular integral equations with kernels of a special form.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.