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Abstract

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ON THE EXISTENCE OF INVARIANT MEASURES ON BOOLEAN ALGEBRAS

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In this note we consider the following problem. Let X be a complete Boolean algebra, and let $\mathfrak{A}$ be a group of its automorphisms.* What properties of the algebra X, on the one hand, and of the collection of automorphisms $\mathfrak{A}$, on the other, guarantee the existence on X of a measure invariant with respect to all automorphisms of the group $\mathfrak{A}$? This problem was partly touched upon by us in the note ([1]), in which one may also find explanations concerning the terms and notation occurring below.

As in ([1]), by a **measure** we mean a real-valued function μ defined on X and possessing the following properties:

1) $\mu(x) > 0$ for $x \neq 0$;
2) $\mu(x+y) = \mu(x) + \mu(y)$ for $x \wedge y = 0$;
3) $\mu(x_n) \downarrow 0$ when $x_n \downarrow 0$;
4) $\mu(1) = 1$.

Properties 2) and 3) together mean the countable additivity of the measure. A function μ nonnegative for all x and possessing properties 2) and 4) will be called a **quasimeasure**. Invariance of a measure or quasimeasure consists in the fact that, for all $x \in X$, $A \in \mathfrak{A}$,

$$\mu(Ax) = \mu(x).$$

Let us now enumerate those properties of the collection of automorphisms $\mathfrak{A}$ which will be important for us.

A_1. For all $x \neq 0$,

$$\sup_{A \in \mathfrak{A}} Ax = 1.$$

(This is an ergodicity property of $\mathfrak{A}$; it means that there are no elements fixed under all A other than 0 and 1.)

A stronger requirement is contained in the condition

A_2. For every $x \neq 0$ there exists a finite number of automorphisms $A_1, A_2, \dots, A_n \in \mathfrak{A}$ such that

$$\sup_{k=1,\dots,n} A_k x = 1.$$

Properties A_1 and A_2 say that the group $\mathfrak{A}$ is "sufficiently rich"in automorphisms. We now pass to the enumeration of properties having the opposite character. We shall agree to write $x \simeq y$ if $y = Ax$ for some $A \in \mathfrak{A}$. Further, if there exist representations of x and y in the form of sums of disjoint elements

$$x = \sum_{\alpha} x_{\alpha}, \qquad y = \sum_{\alpha} y_{\alpha},$$

where $x_{\alpha} \simeq y_{\alpha}$, then we shall write $x \sim y$. Here the cardinality of the set of summands may be arbitrary. In the case when the sums are finite, we shall write $x \simeq y$. Finally, we shall agree to denote by an arrow $\to$ convergence in the order topology (i.e., in the (o)-topology; see ([2])).

B_1. If $x_n \to 0$, then $A_n x_n \to 0$, whatever $A_n \in \mathfrak{A}$ may be ("uniform continuity" of all automorphisms of the group).

B_2. If $x_n \simeq x_{n+1}$, $n = 1, 2, \dots$, and $x_n \wedge x_m = 0$, $n \neq m$, then

$$x_1 = x_2 = \dots = 0.$$

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* As usual, by an automorphism we mean a one-to-one mapping of X onto itself that preserves order. The necessary information from the theory of partially ordered sets is contained, for example, in the monograph ([2]).

B_3. If $x_n = \sum_{\alpha} x_{n\alpha}$ and $x_n \to 0$, then for any $A_{n\alpha} \in \mathfrak{A}$ one has

$$\sup_{\alpha} A_{n\alpha} x_{n\alpha} \to 0,$$

where the index α ranges over an arbitrary set depending on n.

$B_3{}^{-}$. If $x_n \sim x_{n+1}$, $n = 1, 2, \dots$, and $x_n \wedge x_m = 0$, $n \neq m$, then

$$x_1 \sim x_2 = \dots = 0.$$

$B_3{}''$. If $x_n \simeq x_{n+1}$, $n = 1, 2, \dots$, and $x_n \wedge x_m = 0$, $n \neq m$, then

$$x_1 = x_2 = \dots = 0.$$

B_4. The relations $x < y$ and $x \sim y$ are incompatible.

B_5. For every $x \neq 0$ there exists an invariant quasimeasure μ such that

$$\mu(x) > 0.$$

diagram of implications among B_1, B_2, B_3, $B_3{}^-$, $B_3{}''$, B_4, B_5, B_6

Figure 1: diagram of implications among B_1, B_2, B_3, $B_3{}^-$, $B_3{}''$, B_4, B_5, B_6

diagrams of implications involving B_1 & K, B_2 & K, A_1 & B_3, A_1 & B_4, A_2 & K, m, km, B_3, B_5

Figure 2: diagrams of implications involving B_1 & K, B_2 & K, A_1 & B_3, A_1 & B_4, A_2 & K, m, km, B_3, B_5

B_6. If

$$\sum_k x_k \leq 1,$$

then for any $A_k \in \mathfrak{A}$ one has

$$\sum_k A_{kx} k < +\infty$$

in the extended K-space "built over" the algebra X.

Finally, let us add to the listed conditions one more, of a purely algebraic character.

K. On the group $\mathfrak{A}$ there exists a Banach mean. This condition is satisfied, for example, by every commutative or even solvable group (see on this matter ([3]); ([4]), p. 141).

Theorem 1. *The relation among the conditions* B_1, ⋯, K *is described by the following diagram:*

Here a heavy arrow means "implies in every algebra," while a light arrow means "implies in a regular algebra."

We shall agree to denote by m the assertion: "there exists an invariant measure" ; and by km the assertion: "there exists an invariant quasimeasure possessing property 1)."

Theorem 2. *The relation of the conditions* A_1, ⋯, B_6 *to the assertions* m, km *is described by the following diagrams:*

Here the heavy and light arrows have the former meaning; a dotted arrow means "implies in a normalized algebra" (i.e. in an algebra on which there exists some measure with properties 1)–4)); & is the logical sign of conjunction.

Let us note the following special cases.

a) The fact that conditions A_1 and B_3 together imply the existence of an invariant measure constituted the content of Theorem 1 of the paper ([1]). Algebras with a group of automorphisms having properties A_1 and B_3 were called completely homogeneous in ([1]). There the general form of such an algebra was also established. Theorem 2 shows, that in this definition B_3 may be replaced by B_4 (although the equivalence of B_3 and B_4 was established by us in Theorem 1 only for a regular algebra).

b) Theorem 2 shows that in a normed algebra the existence of an invariant measure is ensured by conditions K and B_2. In particular, when the group consists of powers of a single automorphism, we obtain, as a consequence, the Hajian–Kakutani theorem on an invariant measure (see ([5])).

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Note: Figure translations are in progress. See original paper for figures.