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Abstract

Full Text

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Weakly Completely Continuous Spectra of Locally Convex Spaces

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As B. M. Makarov showed ([1]), limits of countable spectra of locally convex spaces with weakly completely continuous mappings have properties largely analogous to the properties of limits of countable spectra with completely continuous mappings, established earlier by Sebastião e Silva ([2]) and D. A. Raikov ([3]). On the other hand, D. A. Raikov ([4]) also studied limits of arbitrary (in general uncountable) spectra with completely continuous mappings (called by him completely continuous spectra). In the present note analogous properties, as well as some specific properties, are established for limits of arbitrary spectra of locally convex spaces with weakly completely continuous mappings.

For definitions of the general concepts of the theory of topological linear spaces used here, see ([5]), and for those relating to spectra of locally convex spaces, see ([4]).

Definition 1. A linear mapping φ of a locally convex space E into a locally convex space F is called **weakly completely continuous** if, for some neighborhood of zero $U \subset E$, $\varphi(U)$ is weakly bicompact in F.

1. Let $\{E_\alpha, \pi_\alpha^\beta\}$ be the inverse spectrum of a family of locally convex spaces E_α and continuous linear mappings $\pi_\alpha^\beta : E_\beta \to E_\alpha$, and let

$$E = \varprojlim \{E_\alpha, \pi_\alpha^\beta\}$$

be its limit (projective limit).

Definition 2. An inverse spectrum $\{E_\alpha, \pi_\alpha^\beta\}$ is called **weakly completely continuous** if for every α there exists $\beta > \alpha$ such that $\pi_\alpha^\beta : E_\beta \to E_\alpha$ is weakly completely continuous. A locally convex space representable as the limit of a weakly completely continuous nondegenerate ([4]) inverse spectrum is called a **space of type** (W).

Theorem 1. *A space of type* (W) *is complete and semireflexive.*

Theorem 2. *Every space of type* (W) *is representable as the limit of a weakly completely continuous inverse spectrum of Banach spaces.*

Theorem 3. *A closed subspace of a space of type* (W)*, a complete quotient space of a space of type* (W)*, the product of any family of spaces of type* (W)*,* *the projective limit of any family of spaces of type (W), and the inductive limit of a sequence of spaces of type (W), if it is a complete separated space, are also spaces of type (W).*

In ([6]), in studying the weak complete continuity of a linear mapping and of its adjoint, a class of spaces (III) was considered. The relation between spaces of class (III) and spaces of type (W) is given by the following theorem:

Theorem 4. *For a complete semireflexive locally convex space E, the following properties are equivalent:*

1°. *E is a space of class (III).*

2°. *E is of type (W), and the topology of E coincides with the Mackey topology $\tau(E, E')$.*

Remark 1. There exist spaces of type (W) that do not belong to class (III).

Example. Let E be an infinite-dimensional reflexive Banach space. Introduce on E the topology of $c(E, E')$-compact convergence, i.e.

the topology of uniform convergence on all compact subsets of E'; denote the resulting space by $(E, c(E, E'))$. $(E, c(E, E'))$ is of type (W). But $c(E, E')$ is weaker than $\tau(E, E')$, since in E' there exists a weakly bicompact but not bicompact set (the unit ball). Consequently (Theorem 4), $(E, c(E, E'))$ does not belong to class (III).

In ([6]) it was indicated that a quasinormed space ([7]) belongs to class (III) if and only if its topology majorizes the topology T of uniform convergence on all convex $\sigma(E', E'')$-bicompact subsets of E'. However:

Theorem 5. *Every complete semireflexive quasinormed space is a space of type (W).*

Proof. *Let E be a space of type (DF) ([7]), endowed with the Mackey topology. In order that E be of type (W), it is necessary and sufficient that E be reflexive and complete.*

2. Let $\{E_\alpha, \pi^\alpha_\beta\}$ be a direct spectrum of a family of locally convex spaces E_α and continuous linear mappings $\pi^\alpha_\beta : E_\alpha \to E_\beta$, and let

$$E = \lim_{\longrightarrow}\{E_\alpha, \pi^\alpha_\beta\}$$

be its limit (inductive limit).

Definition 3. A direct spectrum $\{E_\alpha, \pi^\alpha_\beta\}$ is called **weakly completely continuous** if for every α there is a $\beta > \alpha$ such that $\pi^\alpha_\beta : E_\alpha \to E_\beta$ is weakly completely continuous. A locally convex space representable as the limit of a weakly completely continuous direct spectrum is called a **space of type** (W').

Theorem 6. *Every space of type (W') is representable as the limit of a weakly completely continuous direct spectrum of Banach spaces.*

Analogously to [4], one can show that the class of spaces (W') coincides with the class of spaces of type (β) (a space of type (β) is an inductive limit of Banach spaces).

3. **Theorem 7.** *If E is a space of type (W), then E' is of type (W'). Conversely, every E of type (W') is the strong dual of some space of type (W).*

In [6] spaces of class (II) were studied. It can be shown that a countable inductive limit of spaces of class (II) (even of Banach spaces) need not be a space of class (II). However:

Theorem 8. *Every space of type (W') representable as the limit of a weakly completely continuous direct countable spectrum belongs to class (II).*

Definition 4. We shall say that a locally convex space E is of **type** (K) if, for every convex weakly bicompact set $K_1 \subset E$, there is a convex weakly bicompact set $K_2 \subset E$ such that K_1 is weakly bicompact in the Banach space E_{K_2} (this is the vector space that is the linear hull of the set K_2, endowed with the normed topology whose unit ball is the balanced hull of the set K_2).

Theorem 9. *If E is a space of type (W) with the Mackey topology, then E' is of types (W') and (K). Conversely, every E of types (W') and (K) is the strong dual of some space of type (W), endowed with the Mackey topology.*

Theorem 10. *The strong dual of a space E of type (W') is of type (W) if and only if E is of type (K) and reflexive.*

Remark 2. There exist complete semireflexive spaces with the Mackey topology which are not reflexive. Namely, there exist spaces of type (W) with the Mackey topology, but not barrelled. For example, consider the space l^1 of summable sequences. It can be shown that l^1 is a space of types (K) and (W'). Then (Theorem 9) l^1 is the strong dual of a space E of type (W), whose topology is the Mackey topology. But E is not barrelled; for otherwise l^1 would be a reflexive space (Theorem 10), which is not the case. Here E is even

space of type $(\overline{S})$ [4], so that this example shows that there may be spaces of type $(\overline{S})$ with the Mackey topology, but not of type (M) (although a metrizable space of type $(\overline{S})$ is always of type (M) [7]).

4. From Theorem 9 it follows that the space of types (W') and (K) belongs to class (II) [6]. In addition, for spaces of type (K) the following holds.

Theorem 11. *For a locally convex space E, the following assertions are equivalent:*

1°. *E is barrelled and of type (K).*

2°. *E is the strong dual of some semi-reflexive space of class (III).*

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Note: Figure translations are in progress. See original paper for figures.