DIFFRACTION OF A PLANE LONGITUDINAL WAVE BY A CIRCULAR CYLINDER

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Fig. 1

Figure 1: Fig. 1

Abstract

Full Text

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DIFFRACTION OF A PLANE LONGITUDINAL WAVE BY A CIRCULAR CYLINDER

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We consider the incidence of a stationary plane longitudinal wave with potential $\varphi_0(r,t) = e^{i(k_1 r\cos\vartheta - \omega t)}$ on a stress-free cylinder of radius a, situated in an elastic medium (Fig. 1). The solution of analogous problems in the form of series was given in ([1]), but these series converge well only when $a \ll 2\pi/k$, i.e., when the cylinder diameter is small in comparison with the wavelength. In ([2,3]) a fairly general method was proposed for solving problems of scattering of elastic waves by curvilinear surfaces. However, the phenomena that arise in this case (in accordance with Kirchhoff' s principle) are considered only in the illuminated region. In the present note we use the method proposed by Watson ([4]) and further developed in ([5–7]), etc., for problems of acoustics and electrodynamics –the method of transforming sums of series into contour integrals. We then investigate the displacements arising on the cylinder, expressed through contour integrals in the regions of "light," "shadow," and "penumbra" for the case $k_1 a$ and $k_2 a \gg 1$, where k_1 and k_2 are the wave numbers of longitudinal and transverse waves. The solution for the potentials of longitudinal and transverse waves outside the cylinder is written in the form ([1])

Fig. 1

$$\varphi_0 + \varphi_1 = \sum_0^\infty \varepsilon_n (i)^n \left[-\frac{\Delta_{J_n}}{\Delta_n} H_n^{(1)}(k_1 r) + J_n(k_1 r) \right] \cos n\vartheta,$$

$$\psi_1 = \frac{4i}{\pi} \sum_0^\infty \varepsilon_n (i)^n \frac{n(y^2 + 2 - 2n^2)}{\Delta_n} H_n^{(1)}(k_2 r) \sin n\vartheta, \qquad (1)$$

where $H_n^{(1)}(\rho)$ are Hankel functions of the first kind; $J_n(\rho)$ are Bessel functions;

$$\varepsilon_0 = 1; \qquad \varepsilon_n = 2 \ (n = 1, 2, ...); \qquad k_1 a = x; \quad k_2 a = y;$$

Fig. 2

Figure 2: Fig. 2

$$\Delta_n = \left[4n^2 - (2n^2 - y^2)^2\right] H_n^{(1)}(x) H_n^{(1)}(y) + 4xy(n^2 - 1) H_n^{(1)\prime}(x) H_n^{(1)\prime}(y) -$$

$$-2y^2 \left[x H_n^{(1)}(y) H_n^{(1)\prime}(x) + y H_n^{(1)}(x) H_n^{(1)\prime}(y)\right]; \tag{2}$$

Δ_{J_n} is analogous to Δ_n, with $H_n^{(1)}(x)$ replaced by $J_n(x)$.

Let us study the displacements caused by the incidence of the wave at the boundary of the cylinder:

$$u = \frac{2y^2 i}{\pi a} \sum_0^\infty \varepsilon_n \frac{(y^2 - 2n^2) H_n^{(1)}(y) + 2y H_n^{(1)\prime}(y)}{\Delta_n} \cos n\vartheta \, e^{in\pi/2},$$

$$v = -\frac{4y^2 i}{\pi a} \sum_0^\infty \varepsilon_n \frac{n\left[y H_n^{(1)\prime}(y) - H_n^{(1)}(y)\right]}{\Delta_n} \sin n\vartheta \, e^{in\pi/2}. \tag{3}$$

It is known that if the quantities $k_1 a$ and $k_2 a \gg 1$, then these series converge very slowly and are unsuitable for practical use. Therefore we use Watson' s method of transforming the sums of the series (3) into integrals over contours of the complex ν-plane. To this end, consider the functions of the complex variable

$$f_1(\nu) = \frac{(y^2 - 2\nu^2) H_\nu^{(1)}(y) + 2y H_\nu^{(1)\prime}(y)}{\Delta_\nu} \frac{\cos \nu\vartheta}{\sin \nu\pi} e^{-i\nu\pi/2}, \tag{4a}$$

$$f_2(\nu) = \frac{\nu\left[y H_\nu^{(1)\prime}(y) - H_\nu^{(1)}(y)\right]}{\Delta_\nu} \frac{\sin \nu\vartheta}{\sin \nu\pi} e^{-i\nu\pi/2}. \tag{46}$$

Then the displacements at the boundary of the cylinder are represented in the form

$$u = u_1 + u_{1R} = -\frac{2y^2}{\pi a} \int_C f_1(\nu)\, d\nu - \frac{4y^2 i}{a} \operatorname{Res}\left[f_1(\nu)\right]_{\nu^*}, \tag{5}$$

$$v = v_1 + v_{1R} = \frac{4y^2}{\pi a} \int_C f_2(\nu)\, d\nu + \frac{8y^2 i}{a} \operatorname{Res}\left[f_2(\nu)\right]_{\nu^*},$$

Fig. 2

where C is a loop around the real semiaxis, ν^* is the real positive root of the equation $\Delta_\nu = 0$ (Fig. 2). $f_1(\nu)$ and $f_2(\nu)$ are odd analytic functions of ν, meromorphic in the right half-plane, having poles only in the first quadrant and on the real axis.

On the basis of what has been said, we transform the path of integration C and split it into parts, as shown in Fig. 2, where A, B, D, E, F are points of a circle of infinitely large radius; $\nu_1, \dots, \nu_k$ are zeros of the function Δ_ν lying in the first quadrant.

The integrals over portions of infinite circles, as can readily be shown using the properties of the functions $f_1(\nu)$ and $f_2(\nu)$, tend to zero. The integrals over BD, owing to the evenness of the integrand functions, vanish. Thus,

$$u_1 = -\frac{2y^2}{\pi a} \int_E^F \frac{(y^2 - 2\nu^2) H_\nu^{(1)}(y) + 2y H_\nu^{(1)\prime}(y)}{\Delta_\nu} \frac{\cos \nu\vartheta}{\sin \nu\pi} e^{-i\nu\pi/2} \, d\nu,$$

$$v_1 = \frac{4y^2}{\pi a} \int_E^F \frac{\nu}{\Delta_\nu} \left[y H_\nu^{(1)\prime}(y) - H_\nu^{(1)}(y) \right] \frac{\sin \nu\vartheta}{\sin \nu\pi} e^{-i\nu\pi/2} \, d\nu. \qquad (6)$$

The integrals over EF can be evaluated through the sum of the residues of the integrand functions at the zeros $\nu_1, \dots, \nu_k, \dots$, lying in the first quadrant:

$$u_1 = -\frac{4y^2 i}{a} \sum_{k=1}^{\infty} \frac{(y^2 - 2\nu_k^2) H_{\nu_k}^{(1)}(y) + 2y H_{\nu_k}^{(1)\prime}(y)}{(\partial \Delta_\nu / \partial \nu)_{\nu_k}} \frac{\cos \nu_k \vartheta}{\sin \nu_k \pi} e^{-i\nu_k \pi/2}, \qquad (7)$$

$$v_1 = \frac{8y^2 i}{a} \sum_{k=1}^{\infty} \frac{\nu_k}{(\partial \Delta_\nu / \partial \nu)_{\nu_k}} \left[y H_{\nu_k}^{(1)\prime}(y) - H_{\nu_k}^{(1)}(y) \right] \frac{\sin \nu_k \vartheta}{\sin \nu_k \pi} e^{-i\nu_k \pi/2}.$$

From the subsequent investigation it will be seen that these series converge in the region where $|\vartheta| < \pi/2$, i.e., in the shadow.

For the illuminated region the integrals (6) should be transformed, and then the displacements in the illuminated region are represented in the form of the sum of three terms:

$$u = -\frac{2y^2}{\pi a} \int_E^F \frac{(y^2 - 2\nu^2) H_\nu^{(1)}(y) + 2y H_\nu^{(1)\prime}(y)}{\Delta_\nu} \frac{\cos \nu(\pi - \vartheta)}{\sin \nu\pi} e^{i\nu\pi/2} \, d\nu +$$

$$+ \frac{2y^2 i}{\pi a} \int_E^F \frac{(y^2 - 2\nu^2) H_\nu^{(1)}(y) + 2y H_\nu^{(1)\prime}(y)}{\Delta_\nu} e^{i\nu(\pi/2 - \vartheta)} \, d\nu + u_{1R}, \qquad (8a)$$

$$v = -\frac{4y^2}{\pi a} \int_E^F \frac{\nu \left[y H_\nu^{(1)\prime}(y) - H_\nu^{(1)}(y) \right]}{\Delta_\nu} \frac{\sin \nu(\pi - \vartheta)}{\sin \nu\pi} e^{i\nu\pi/2} \, d\nu +$$

$$+\frac{4y^2}{\pi a}\int_E^F \frac{\nu\left[yH_\nu^{(1)\prime}(y)-H_\nu^{(1)}(y)\right]}{\Delta_\nu}e^{i\nu(\pi/2-\vartheta)}\,d\nu+v_{1R}. \tag{86}$$

The first integrals, analogously to the preceding one, are evaluated through the sum of residues at the zeros of Δ_ν; the second can be transformed to an integral over BD

$$u=u_2+u_3+u_R=$$

$$=-\frac{4y^2i}{a}\sum_{k=1}^{\infty}\frac{(y^2-2\nu_k^2)H_{\nu_k}^{(1)}(y)+2yH_{\nu_k}^{(1)\prime}(y)}{(\partial\Delta_\nu/\partial\nu)_{\nu_k}}\frac{\cos\nu_k(\pi-\vartheta)}{\sin\nu_k\pi}e^{i\nu_k\pi/2}-$$

$$-\frac{2y^2i}{\pi a}\int_B^D\frac{(y^2-2\nu^2)H_\nu^{(1)}(y)+2yH_\nu^{(1)\prime}(y)}{\Delta_\nu}e^{i\nu(\pi/2-\vartheta)}\,d\nu+u_R; \tag{9}$$

$$v=v_2+v_3+v_R=-\frac{8y^2i}{a}\sum_{k=1}^{\infty}\frac{\nu_k\left[yH_{\nu_k}^{(1)\prime}(y)-H_{\nu_k}^{(1)}(y)\right]}{(\partial\Delta_\nu/\partial\nu)_{\nu_k}}\frac{\sin\nu_k(\pi-\vartheta)}{\sin\nu_k\pi}e^{i\nu_k\pi/2}-$$

$$-\frac{4y^2}{\pi a}\int_B^D\frac{\nu\left[yH_\nu^{(1)\prime}(y)-H_\nu^{(1)}(y)\right]}{\Delta_\nu}e^{i\nu(\pi/2-\vartheta)}\,d\nu+v_R,$$

where

$$u_R=-\frac{4y^2i}{a}\frac{(y^2-2\nu^{*2})H_{\nu^*}^{(1)}(y)+2yH_{\nu^*}^{(1)\prime}(y)}{(\partial\Delta_\nu/\partial\nu)_{\nu^*}}\frac{\cos\nu^*(\pi-\vartheta)}{\sin\nu^*\pi}e^{i\nu^*\pi/2},$$

$$v_R=-\frac{8y^2i}{a}\frac{\nu^*\left[yH_{\nu^*}^{(1)\prime}(y)-H_{\nu^*}^{(1)}(y)\right]}{(\partial\Delta_\nu/\partial\nu)_{\nu^*}}\frac{\sin\nu^*(\pi-\vartheta)}{\sin\nu^*\pi}e^{i\nu^*\pi/2}. \tag{10}$$

Next, to compute the displacements it is necessary to use the asymptotic form of the Hankel functions for large arguments. For the regions where

$$|x^2-\nu^2|\gg Ax^{4/3},\qquad |y^2-\nu^2|\gg Ay^{4/3},\qquad A\sim 3, \tag{11}$$

one may use for $H_\nu^{(1)}(x)$, $H_\nu^{(1)}(y)$ and their derivatives Debye' s asymptotic formula (8). The equation $\bar\Delta_\nu\equiv\left(H_\nu^{(1)}(x)H_\nu^{(1)}(y)\right)^{-1}\Delta_\nu=0$ in the Lebedev approximation has the form

$$\left(2\frac{\nu^2}{y^2}-1\right)^2-4\frac{x}{y}\frac{\nu^2}{y^2}\sqrt{\frac{\nu^2}{x^2}-1}\sqrt{\frac{\nu^2}{y^2}-1}- \tag{12}$$

$$-\frac{2}{y}\left[\frac{x}{y}\sqrt{\frac{\nu^2}{x^2}-1}+\sqrt{\frac{\nu^2}{y^2}-1}\right]-\frac{4}{y^2}\left[\frac{\nu^2}{y^2}-\frac{x}{y}\sqrt{\frac{\nu^2}{x^2}-1}\sqrt{\frac{\nu^2}{y^2}-1}\right]=0.$$

To accuracy up to y^{-1}, equation (12) is equivalent to the Rayleigh equation, which has a real positive root $\nu^* = \chi y$, where $\chi > 1$ and χ depends on $\varepsilon = k_1/k_2 < 1$. Therefore the residues at the point ν^*, corresponding to the displacement components u_R, v_R, u_{1R}, and v_{1R}, give the displacements in the surface wave of Rayleigh-wave type. The displacement components u_3 and v_3, represented in (9) by integrals over BD, can be evaluated by the saddle-point method:

$$u_3 = \frac{2k_1 i(2\varepsilon^2\sin^2\vartheta-1)\cos\vartheta}{(2\varepsilon^2\sin^2\vartheta-1)^2-4\varepsilon^3\sin^2\vartheta\cos\vartheta\sqrt{1-\varepsilon^2\sin^2\vartheta}}e^{ik_1 a\cos\vartheta},$$

$$v_3 = -\frac{4\varepsilon k_1 i\sin\vartheta\cos\vartheta\sqrt{1-\varepsilon^2\sin^2\vartheta}}{(2\varepsilon^2\sin^2\vartheta-1)^2-4\varepsilon^3\sin^2\vartheta\cos\vartheta\sqrt{1-\varepsilon^2\sin^2\vartheta}}e^{ik_1 a\cos\vartheta}. \tag{13}$$

From (14) we see that u_3 and v_3 give the geometrical components of the displacement field.

Debye' s asymptotic formula for the function $H_\nu^{(1)}(x)$ will be inapplicable: (a) near the boundary of the shadow, since there $|\nu_0^2 - x^2| \ll Ax^{4/3}$, i.e. $|\cos(\pi - \vartheta)| \sim (k_1 a)^{-1/3}$, and (b) in the calculation of the smallest roots of the equation $\Delta_\nu = 0$, since they lie inside the Fock circle. In these cases for the Hankel function

for $H_\nu^{(1)}(y)$ one can as before use Debye' s asymptotic formula, and for $H_\nu^{(1)}(x)$ the more accurate Fock-Hankel asymptotic formula.

For calculating the diffraction components of the displacements u_2, v_1 and u_2, v_2, it is usually sufficient to take several first terms of the series in ν_k. For $\nu_k = x + (x/2)^{1/3}t_k$, lying inside the Fock region, $t_k(x/2)^{-2/3} \ll 1$, where t_k is a root of the equation $w'(t) - q_1 w(t) = 0$, and $w(t)$ is the Airy function. Then

$$u_1 = \frac{k_1 i\sqrt{\pi}}{2}\frac{1-2\varepsilon^2}{\varepsilon^3\sqrt{1-\varepsilon^2}}\sum_k \frac{1}{w(t_k)(t_k-q_1^2)}\frac{\cos\nu_k\vartheta}{\sin\nu_k\pi}e^{-i\nu_k\pi/2},$$

$$v_1 = \frac{k_1\sqrt{\pi}}{\varepsilon^2}\sum_k \frac{1}{w(t_k)(t_k-q_1^2)}\frac{\sin\nu_k\vartheta}{\sin\nu_k\pi}e^{-i\nu_k\pi/2}, \tag{14}$$

where q_1 is a constant quantity equal to $-i(x/2)^{1/3}(2\varepsilon^2-1)^2/4\varepsilon^3\sqrt{1-\varepsilon^2}$.

The equation $w'(t) - q_1 w(t) = 0$ is studied in detail in work (5). The investigation shows that the series (16) converge for $|\vartheta| < \pi/2$, i.e., they can be used to calculate displacements only in the shadow region. Analogous series are obtained for the displacements u_2 and v_2 in the illuminated region:

$$u_2 = \frac{k_1 i\sqrt{\pi}}{2}\frac{1-2\varepsilon^2}{\varepsilon^3\sqrt{1-\varepsilon^2}}\sum_k \frac{1}{w(t_k)(t_k-q_1^2)}\frac{\cos\nu_k(\pi-\vartheta)}{\sin\nu_k\pi}e^{i\nu_k\pi/2},$$

$$v_2 = -\frac{k_1\sqrt{\pi}}{\varepsilon^2}\sum_k \frac{1}{w(t_k)(t_k-q_1^2)}\frac{\sin\nu_k(\pi-\vartheta)}{\sin\nu_k\pi}e^{i\nu_k\pi/2}. \qquad (15)$$

The series (15) converge everywhere, i.e., they are also suitable in the penumbra region. To calculate the total displacement field in the penumbra, it remains to indicate how to calculate the integrals over BD for $\vartheta \sim \pi/2$, since the saddle-point approximation is not applicable there owing to the closeness of ν_0 to $k_1 a$. Using the Hankel–Fock approximation, we obtain

$$u_3 = \frac{k_1 i\exp\{ik_1 a(\pi/2-\vartheta)\}}{4\varepsilon^3\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{1-2\varepsilon^2\left[1+(x/2)^{-2/3}t/2\right]^2}{\left[1+(x/2)^{-2/3}t/2\right]^2\sqrt{1-\varepsilon^2\left[1+(x/2)^{-2/3}t/2\right]^2}}\times\frac{\exp\{i(x/2)^{1/3}t(\pi/2-\vartheta)\}}{w'(t)-qw(t)}$$

$$v_3 = \frac{k_1 i\exp\{ik_1 a(\pi/2-\vartheta)\}}{2\varepsilon^2\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{\exp\{i(x/2)^{1/3}t(\pi/2-\vartheta)\}}{\left[1+(x/2)^{-2/3}t/2\right]}\frac{dt}{w'(t)-qw(t)}. \qquad (16)$$

The integrals (16) can be calculated by quadrature formulas, replacing, in the t-plane, integration over BD by integrals over the broken line Γ_1, going from $\infty e^{2i\pi/3}$ to 0 along the straight line $\arg t = 2\pi/3$, and from 0 to ∞ along the real axis. The resulting integrals with real limits are not difficult to compute by means of the tables given in (5).

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Note: Figure translations are in progress. See original paper for figures.