Mathematics

A. G. SIGALOV

1964

SovietRxiv

Abstract

Full Text

Mathematics

A. G. SIGALOV

A NEW ALGORITHM IN THE PERTURBATION THEORY OF THE CONTINUOUS SPECTRUM

(Presented by Academician V. I. Smirnov on 16 III 1964)

No. 1. Let H be a self-adjoint operator with simple spectrum, acting in a Hilbert space $\mathfrak{H}$; $E_\Delta^H = E_\Delta$ is its spectral function; ψ is a generating element; $\sigma(\Delta) = (E_\Delta \psi, \psi)$. Put $A \in \mathfrak{M}(H)$, if $D_A \supset D_H$, A is symmetric on D_H, and the following hold: 1) $|A\varphi|^2 \leq C_1 |H\varphi|^2 + C_2 |\varphi|^2$ for all $\varphi \in D_H$; 2) if

$$G_\eta^N = \{(\alpha, \beta),\ |\alpha - \beta| \geq \eta,\ |\alpha| \leq N,\ |\beta| \leq N\},$$

then for any $\eta > 0$, $N > 0$ there is a $C = C(N, \eta) > 0$ such that from $\Delta\alpha \times \Delta\beta \subset G_\eta^N$ it follows that

$$|(AE_{\Delta\alpha}\psi, E_{\Delta\beta}\psi)| \leq C |E_{\Delta\alpha}\psi|^2 |E_{\Delta\beta}\psi|^2;$$

$\Delta\alpha, \Delta\beta$ are always intervals open on the right. From 1) it follows: 3) $|(AE_{\Delta\alpha}\psi, E_{\Delta\alpha}\psi)| \leq C_3 |E_{\Delta\alpha}\psi|^2$, if $\Delta\alpha \in [-N, N]$, $C_3 = C_3(N)$.

Let $m(E)$ $(\sigma_2(E))$ be the measure, defined on the Borel field of sets $\mathfrak{B}_m$ $(\mathfrak{B}_{\sigma_2})$, which is the minimal extension of the interval functions

$$m(\Delta\alpha, \Delta\beta) = (AE_{\Delta\alpha}\psi, E_{\Delta\beta}\psi)$$

(respectively, of the function $\sigma(\Delta\alpha) \cdot \sigma(\Delta\beta)$), $\Delta\alpha \times \Delta\beta \in G_\eta^N$ (^{1}). We have $\mathfrak{B}_m \supset \mathfrak{B}_{\sigma_2}$; $m(E)$ is absolutely continuous with respect to $\sigma_2(E)$ (^{1}). By the Radon-Nikodym theorem, on G_η^N a function

$$A_{\alpha\beta}^H = \frac{dm}{d\sigma_2}$$

is defined almost everywhere with respect to σ_2. The function $A_{\alpha\beta}$ does not depend on N or η.

No. 2. Let $A > 0$, $A \in \mathfrak{M}(H)$, and let $\Delta' = [\alpha, \alpha')$ be a finite interval,

$$\Pi = \{\alpha_0 < \alpha_1 < \cdots < \alpha_N = \alpha'\}, \quad \Delta_i = [\alpha_{i-1}, \alpha_i),$$

$$V(\Pi) = \sum (AE_{\Delta_i}\psi, E_{\Delta_i}\psi),$$

$$d(\Pi) = \max_i(\alpha_i - \alpha_{i-1}).$$

Put

$$m_1(\Delta') = \lim_{d(\Pi)\to 0} V(\Pi).$$

The interval function $m_1(\Delta)$ is additive and absolutely continuous with respect to $\sigma(\Delta)$. Extending them to the corresponding measures $m_1(E)$, $E \in \mathfrak{B}_{m_1}$, $\sigma(E)$, $E \in \mathfrak{B}_\sigma$, and using the Radon-Nikodym theorem, define

$$A_\alpha^H = \frac{dm_1}{d\sigma}.$$

If: 1) $\sigma(\alpha)$ is absolutely continuous and 2) $C(N, \eta)$ does not depend on η, then A_α^H can be defined as

$$\lim_{\Delta\to\alpha} \frac{1}{\sigma(\Delta)}(AE_\Delta\psi, E_\Delta\psi)$$

independently of the assumption $A > 0$.

The pair of functions $A_{\alpha\beta}^H$, A_α^H will be called the diagonally singular matrix (DS-matrix) of the operator A relative to H.

No. 3. For

$$\varphi = \int c(\alpha)\, dE_\alpha\psi \in D_H, \qquad \chi = \int e(\alpha)\, dE_\alpha\psi \in \mathfrak{H}$$

with continuous $c(\alpha)$, $e(\beta)$, the equality holds

$$(A\varphi, \chi) = \iint c(\alpha)\overline{e(\beta)}\, A_{\alpha\beta}\, d\sigma(\alpha)\, d\sigma(\beta) + \int c(\alpha)\overline{e(\alpha)}\, A_\alpha\, d\sigma(\alpha). \tag{B}$$

Under assumptions 1) and 2) ($A > 0$ is not assumed), the integrals on the right can be understood in the Lebesgue-Stieltjes sense. Without these assumptions, equality (B) requires a generalization of the concept of integral. It can be given starting from such φ, χ for which $c(\alpha)\overline{e(\beta)} = 0$ outside a rectangle $\Delta\alpha \times \Delta\beta$ not intersecting the straight line $\alpha = \beta$, and using property 3) of No. 1.

No. 4. If $A \in \mathfrak{M}(H)$ satisfies one of the conditions ensuring equality (B), then there exists one and only one decomposition $A = B + C$ possessing the properties: a) $B, C \in \mathfrak{M}(H)$; b) $B_\alpha = 0$; c) $CH = HC$. The bilinear form of the operator B is defined by the double integral of the right-hand side of equality (B), while the bilinear form of the operator C is defined by the single integral from (B).

No. 5. Let $H^\varepsilon = H^0 + \varepsilon W$ be bounded self-adjoint operators with simple spectrum and common generating element ψ, $W \in \mathfrak{M}(H^0)$; $W_{\alpha\beta}^\varepsilon = W_{\alpha\beta}^{H^\varepsilon}$, $W_\alpha^\varepsilon = W_\alpha^{H^\varepsilon}$, $\sigma'(\varepsilon, \alpha) = \dfrac{\partial}{\partial\varepsilon}(E_\alpha^{H^\varepsilon}\psi, \psi)$ are continuous, $|\varepsilon| \leqslant \varepsilon_0$.

Then, for any continuous $c(\varepsilon, \alpha)$, $e(\varepsilon, \beta)$,

$$\varphi = \int c(\varepsilon, \alpha)\, dE_\alpha^\varepsilon\psi,$$

$$\chi = \int e(\varepsilon, \beta)\, dE^{\varepsilon}_{\beta} \psi$$

the following equality holds:

$$\frac{\partial}{\partial \varepsilon}(E^{\varepsilon}_{\nu}\varphi, \chi) = \iint g_{\nu}(\alpha, \beta)\, W^{\varepsilon}_{\alpha\beta} c(\varepsilon, \alpha)\overline{e(\varepsilon, \beta)}\, d\sigma(\varepsilon, \alpha)\, d\sigma(\varepsilon, \beta) + \sigma'(\varepsilon, \nu) c(\varepsilon, \nu)\overline{e(\varepsilon, \nu)} W^{\varepsilon}_{\nu},$$

where

$$g_{\nu}(\alpha, \beta) = \begin{cases} 0, & \text{if } (\alpha - \nu)(\beta - \nu) > 0, \\ -|\alpha - \beta|^{-1}, & \text{if } (\alpha - \nu)(\beta - \nu) < 0. \end{cases}$$

The proof is based on the following assertions, formulated, for simplicity of notation, for $H > 0$.

If

$$K_t(r, \nu) = \frac{r(r - \nu \cos t)}{r^2 - 2r\nu \cos t + \nu^2}, \qquad \widetilde{E}^{\varepsilon}_{r} = \frac{1}{2}\{E^{\varepsilon}_{r+0} + E^{\varepsilon}_{r}\},$$

$$\widetilde{E}^{\varepsilon,\delta}_{r} = \frac{1}{\pi}\int_{\delta}^{\pi} K_t(r, H)\, dt,$$

then:

a) $\widetilde{E}^{\varepsilon,\delta}_{r} \to \widetilde{E}^{\varepsilon}_{r}$ strongly;

b)

$$\frac{\partial \widetilde{E}^{\varepsilon,\delta}_{r}}{\partial \varepsilon} = \frac{r}{\pi}\int_{\delta}^{\pi} A^{-1} S A^{-1}\, dt,$$

where

$$A = r^2 E - 2r \cos t\, H^{\varepsilon} + (H^{\varepsilon})^2,$$
$$S = (r^2 W + H^{\varepsilon} W H^{\varepsilon}) \cos t - r(H^{\varepsilon} W + W H^{\varepsilon});$$

c)

$$\frac{\partial}{\partial \varepsilon}(\widetilde{E}^{\varepsilon,\delta}_{r}\varphi, \chi) \to \frac{\partial}{\partial \varepsilon}(E^{\varepsilon}_{r}\varphi, \chi) \quad (\delta \to 0).$$

d) and b) do not require simplicity or continuity of the spectrum.

No. 6. Let $c(\varepsilon, \alpha)$, $e(\varepsilon, \alpha)$, $\rho(\varepsilon, \alpha) = d\sigma(\varepsilon, \alpha)/d\alpha$, $W^{\varepsilon}_{\alpha\beta}$ satisfy a Hölder condition in the variables α, (α, β), $|\alpha| \leqslant M$, $|\beta| \leqslant M$, $|\varepsilon| \leqslant \varepsilon_0$. Put $F(\varepsilon, \nu) = (E^{\varepsilon}_{\nu}\varphi, \chi)$, where $E^{\varepsilon}_{\nu} = E^{H^{\varepsilon}}_{\nu}$. Then

$$\lim_{\Delta\nu \to 0} \frac{1}{2\Delta\nu}\left\{\frac{\partial F(\nu + \Delta\nu, \varepsilon)}{\partial \varepsilon} - \frac{\partial F(\nu - \Delta\nu, \varepsilon)}{\partial \varepsilon}\right\} =$$

$$= -\int \frac{W^{\varepsilon}_{\alpha\nu}}{\alpha - \nu} c(\varepsilon, \alpha)\overline{e(\varepsilon, \nu)}\, d\sigma(\varepsilon, \alpha)\rho(\varepsilon, \nu) - \int \frac{W^{\varepsilon}_{\nu\beta}}{\beta - \nu} c(\varepsilon, \nu)\overline{e(\varepsilon, \beta)}\, d\sigma(\varepsilon, \beta)\rho(\varepsilon, \nu).$$

The integrals on the right-hand side are understood in the sense of the principal value. The limit exists uniformly with respect to ν in every closed interval in which $\rho(\varepsilon, \nu) > 0$. If the exponent and the constant multiplier in the Hölder condition do not depend on ε, $|\varepsilon| \leqslant \varepsilon_0$, then the limit exists uniformly with respect to ε. It follows from this that $\rho(\varepsilon, \alpha)$ and $c(\varepsilon, \alpha)$ satisfy the equations

$$\frac{\partial \rho(\varepsilon, \alpha)}{\partial \varepsilon} = -\int \frac{2 \operatorname{Re} W^{\varepsilon}_{\alpha\nu}}{\alpha - \nu} \rho(\varepsilon, \alpha)\, d\alpha\, \rho(\varepsilon, \nu), \tag{1}$$

$$\frac{\partial c(\varepsilon, \nu)}{\partial \varepsilon} = -\int \frac{W^{\varepsilon}_{\alpha\nu}}{\alpha - \nu} [c(\varepsilon, \alpha) - c(\varepsilon, \nu)] \rho(\varepsilon, \alpha)\, d\alpha. \tag{2}$$

no. 7. Put, for $p(\alpha) = p(\alpha_1, \dots, \alpha_n)$,

$$\Delta_i p = |p(\beta) - p(\alpha)| : |\beta - \alpha|^{\gamma},$$

where $\beta = (\alpha_1, \dots, \alpha_{i-1}, \alpha_i + \Delta\alpha_i, \alpha_{i+1}, \dots, \alpha_n)$,

$$\|p\| = \sup |p(\alpha)| + \sum_i \sup |\Delta_i p(\alpha)| + \sum_{j \neq i} \sup |\Delta_i \Delta_j p(\alpha)|. \tag{2}$$

Let $A^{\varepsilon} \in \mathfrak{M}(H)$, $A^{\varepsilon}_{\alpha} = 0$, $\|A^{\varepsilon}_{\alpha\beta}\| < \infty$; $dA^{\varepsilon}/d\varepsilon = A^{\varepsilon}_1 \in \mathfrak{M}(H)$ exist in the sense of strong convergence and $\|(A^{\varepsilon}_1)_{\alpha\beta}\| < \infty$,

$$P_2 = \{(\nu, \mu);\ \rho(\varepsilon, \nu), \rho(\varepsilon, \mu) > 0\}, \qquad P_1 = \{\nu, \rho(\varepsilon, \nu) > 0\}.$$

Then for $(\nu, \mu) \in P_2$

$$\frac{dA^{\varepsilon}_{\nu\mu}}{d\varepsilon} = \left(\frac{dA^{\varepsilon}}{d\varepsilon}\right)_{\nu\mu} + A^{\varepsilon}_{\nu\mu} \left(\Phi^{\varepsilon}_{\nu} + \Phi^{\varepsilon}_{\mu}\right) + (A^{\varepsilon} V^{\varepsilon} - V^{\varepsilon} A^{\varepsilon})^{H^{\varepsilon}}_{\nu\mu}.$$

Here

$$\Phi^{\varepsilon}_{\nu} = \int \frac{W^{\varepsilon}_{\alpha\nu}}{\alpha - \nu}\, d\sigma(\varepsilon, \nu), \qquad \frac{W^{\varepsilon}_{\alpha\beta}}{\alpha - \beta} = V^{\varepsilon}_{\alpha\beta}.$$

The second term on the right-hand side should be understood as the convolution of DC-matrices of multiplier operators. Putting in this equality $A = W$, we obtain the equation

$$\frac{\partial W^{\varepsilon}_{\nu\mu}}{\partial \varepsilon} = W^{\varepsilon}_{\nu\mu} \left(\Phi^{\varepsilon}_{\nu} + \Phi^{\varepsilon}_{\mu}\right) + (W V^{\varepsilon} - V^{\varepsilon} W)^{H^{\varepsilon}}_{\nu\mu}, \tag{3}$$

which expresses $\partial W^{\varepsilon}_{\nu\mu}/\partial \varepsilon$ in terms of $W^{\varepsilon}_{\nu\mu}$ and $\rho(\varepsilon, \nu)$.

no. 8. Put $T^{\varepsilon}_{\nu\mu} = W^{\varepsilon}_{\nu\mu} \rho(\varepsilon, \nu)$. From (1), (3) it follows that

$$\frac{\partial T^{\varepsilon}_{\nu\mu}}{\partial \varepsilon} = A^{\varepsilon}_{\nu\mu} T^{\varepsilon}_{\nu\mu} + B^{\varepsilon}_{\nu\mu}, \qquad A^{\varepsilon}_{\nu\mu} = -\int \frac{T^{\varepsilon}_{\nu\mu}}{\alpha - \mu}\, dx + \int \frac{T^{\varepsilon}_{\nu\mu}}{\alpha - \nu}\, d\alpha,$$

$$B^{\varepsilon}_{\nu\mu} = -\int T^{\varepsilon}_{\nu\alpha} T^{\varepsilon}_{\alpha\mu} \left\{\frac{1}{\alpha - \nu} + \frac{1}{\alpha - \mu}\right\} d\alpha. \tag{4}$$

Let $\|T^0_{\nu\mu}\|$, $\|\rho(0,\nu)\| < \infty$. Equation (4) has a solution $T^\varepsilon_{\nu\mu}$ in the class of functions with finite norm $\|T^\varepsilon_{\nu\mu}\|$, vanishing for $|\alpha|, |\beta| \geq R$, $T^\varepsilon_{\nu\mu}\big|_{\varepsilon=0} = T^0_{\nu\mu}$. From the solution of equation (4) we obtain solutions of equations (1), (3) on P_1, P_2, which at $\varepsilon = 0$ reduce respectively to $\rho(0,\nu)$ and

$$W^0_{\nu\mu} = T^0_{\nu\mu}\rho(0,\nu)^{-1}.$$

no. 9. From equations (1)–(3) it follows that they have first integrals:

$$\Phi(\varepsilon) = \int c(\varepsilon,\nu)\overline{e(\varepsilon,\nu)}\,\rho(\varepsilon,\nu)\,d\nu \equiv c,$$

$$\Phi_1(\varepsilon) = \iint W^\varepsilon_{\nu\mu} c(\varepsilon,\nu)\overline{e(\varepsilon,\mu)}\,\rho(\varepsilon,\nu)\rho(\varepsilon,\mu)\,d\nu d\mu \equiv c_1,$$

$$\Phi_2(\varepsilon) = \int \nu c(\varepsilon,\nu)\overline{e(\varepsilon,\nu)}\,\rho(\varepsilon,\nu)\,d\nu - \varepsilon\Phi_1(\varepsilon) \equiv c_2.$$

Let $\mathfrak{B}$ be the totality of all $\varphi \in \mathfrak{H}$ for which $c(\nu)$ in the expansion

$$\varphi = \int c(\nu)\,dE^0_\nu\psi$$

has finite norm $\|c(\nu)\|$, and $c(\varepsilon,\nu)$ is the solution of equation (2) equal to $c(\nu)$ at $\varepsilon = 0$. Then, for $\varphi, \chi \in \mathfrak{B}$,

$$(W\varphi,\chi) = \Phi_1(\varepsilon).$$

$$(\varphi,\chi) = \Phi(\varepsilon), \qquad (H^\varepsilon\varphi,\chi) - \varepsilon(W\varphi,\chi) = \Phi_2(\varepsilon),$$

for the left- and right-hand sides do not depend on ε and coincide when $\varepsilon = 0$. Hence it follows that

$$(E^\varepsilon_\nu\varphi,\chi) = \int_{\alpha\leq\nu} c(\varepsilon,\alpha)\overline{e(\varepsilon,\alpha)}\rho(\varepsilon,\alpha)\,d\alpha.$$

Thus, the solutions of equations (1)–(3) make it possible to reproduce the spectral expansion of the operator H^ε.

no. 10. For

$$\varphi = \int c(0,\nu)\,dE^0_\nu\psi$$

put

$$U^\varepsilon\varphi = \int c(0,\nu)\rho(0,\nu)^{1/2} \times \rho(\varepsilon,\nu)^{-1/2}\,dE^\varepsilon_\nu\varphi.$$

Then U^ε is a unitary operator and

$$U^\varepsilon E^0_\Delta = E^\varepsilon_\Delta U$$

for any Δ. The notion of a DC-matrix is carried over to unitary operators. From equations (1)–(3) it is not difficult to obtain an equation for $U^{\varepsilon}_{\alpha\beta}$ and an expression for $\partial U^{\varepsilon}_{\alpha\beta}/\partial\varepsilon$ in terms of $V^{\varepsilon}_{\alpha\beta}$ and Φ^{ε}_{ν}.

No. 11. Let $H^{\varepsilon} = \sum_{k=0}^{n} S_k \varepsilon^k$, $S_k \in \mathfrak{M}(H^0)$, $(S_k)_{\alpha} \equiv 0$, $\|(S_k)_{\alpha\beta}\| < \infty$. Put $W^{i,\varepsilon} = d^{i H^{\varepsilon}}/d\varepsilon^{i}$, $(i = 1, 2, \dots, n)$, $W^{0,\varepsilon} = H^{\varepsilon}$, $W^{n+1,\varepsilon} = 0$. Let $W^{i,\varepsilon}_{\alpha\beta}$ be the DC-matrix of the operator $W^{i,\varepsilon}$ relative to $W^{i-1,\varepsilon}$ $(i = 1, 2, \dots, n)$, $E^{i,\varepsilon}_{\alpha}$ the spectral function of $W^{i,\varepsilon}$, ψ_i a generating element, and $\rho_i(\varepsilon, \alpha) = \dfrac{\partial}{\partial\alpha}(E^{i,\varepsilon}_{\alpha}\psi_i, \psi_i)$. The system of equations generalizing, for $n > 1$, equations (1)–(3), has the form

$$\frac{\partial \rho_i(\varepsilon,\alpha)}{\partial\varepsilon} = -\left(\int \frac{2\,\mathrm{Re}\, W^{i+1,\varepsilon}_{\alpha\nu}}{\alpha-\nu}\,\rho_i(\varepsilon,\alpha)\,d\alpha\right)\rho_i(\varepsilon,\nu) \qquad (i = 0, 1, 2, \dots, n-1),$$

$$\frac{\partial c_i(\varepsilon,\alpha)}{\partial\varepsilon} = -\int \frac{W^{i-1,\varepsilon}_{\alpha\nu}}{\alpha-\nu}\,[c_i(\varepsilon,\alpha) - c_i(\varepsilon,\nu)]\rho_i(\varepsilon,\alpha)\,d\alpha,$$

$$\frac{\partial W^{i,\varepsilon}_{\alpha\beta}}{\partial\varepsilon} = W^{i+1,\varepsilon}_{\alpha\beta} + W^{i,\varepsilon}_{\alpha\beta}(\Phi^{i,\varepsilon}_{\nu} + \Phi^{i,\varepsilon}_{\mu}) +$$

$$+(W^{i,\varepsilon}V^{i,\varepsilon} - V^{i,\varepsilon}W^{i,\varepsilon})^{i,\varepsilon}_{\nu\mu} \qquad (i = 1, 2, \dots, n),$$

$$\Phi^{i,\varepsilon}_{\nu} = \int \frac{W^{i,\varepsilon}_{\alpha\nu}}{\alpha-\nu}\rho_{i-1}(\varepsilon,\alpha)\,d\alpha, \qquad V^{i,\varepsilon}_{\alpha\beta} = \frac{W^{i,\varepsilon}_{\alpha\beta}}{\alpha-\beta}.$$

Putting $T^{i,\varepsilon}_{\alpha\beta} = W^{i,\varepsilon}_{\alpha\nu}\rho_{i-1}(\varepsilon,\alpha)$, we obtain the solution of this system, the reconstruction of the spectral expansion of $W^{i,\varepsilon}$ from the expansion of $W^{i,0}$, and the unitary equivalence of the operator $W^{i,\varepsilon}$ to the operator $W^{i,0}$.

Received
11 III 1964

REFERENCES

[1] P. Halmos, *Measure Theory*, IL, 1953.
[2] K. O. Friedrichs, *Math. Ann.*, **115**, No. 2, 249 (1938).

Note: Figure translations are in progress. See original paper for figures.