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Abstract

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A. L. Onishchik

Connections Without Curvature and the de Rham Theorem

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In the paper (^{1}) we proved Theorem 3, which relates the set of one-dimensional cohomology $H^1(X, \mathfrak{G})$ of a manifold X with coefficients in an arbitrary Lie group $\mathfrak{G}$ (or, equivalently, the set of locally constant fibrations with base X and group $\mathfrak{G}$) to differential forms on the manifold X. In the present paper we refine this theorem, at the same time generalizing (in the one-dimensional case) de Rham's theorems on the existence and cohomology of forms with prescribed periods. The role of closed forms here is played by connections with zero curvature in a fiber space; the role of periods, by the holonomy groups of these connections. In conclusion, a complex-analytic analogue of the theory is considered, and, in particular, results of Röhrl (^{2}) concerning the Riemann-Hilbert problem are obtained.

1. Let X be a smooth manifold, $\mathfrak{G}$ a Lie group, and P a principal fiber space with base X and group $\mathfrak{G}$. Suppose that an infinitesimal connection with zero curvature is given in the fibration P. Fix a point $x_0 \in X$ and a point $b_0 \in P$ lying in the fiber over x_0. Then to every piecewise-smooth closed path l on X, issuing from x_0, there corresponds an element $g \in \mathfrak{G}$ of the holonomy group. If l is homotopic to zero, then $g = e$, since for a connection with zero curvature the restricted holonomy group is trivial (^{3}). Therefore we obtain a homomorphism $\omega : \pi_1(X) \to \mathfrak{G}$, where $\pi_1(X)$ is the fundamental group of the manifold X referred to the point x_0. We shall call the homomorphism ω the **holonomy homomorphism** of our connection. Homomorphisms $\omega_1, \omega_2 : \pi_1(X) \to \mathfrak{G}$ are called **conjugate** if

   $$\omega_2(h) = g_0 \omega_1(h) g_0^{-1} \quad (h \in \pi_1(X))$$

   for some $g_0 \in \mathfrak{G}$. When the point b_0 in the fiber over x_0 is changed, the holonomy homomorphism is replaced by a conjugate one.

We set ourselves the following problems:

1) To determine in what case the holonomy homomorphisms corresponding to two connections without curvature are conjugate to one another.

2) To determine which homomorphisms $\pi_1(X) \to \mathfrak{G}$ can serve as holonomy homomorphisms for connections with zero curvature in P.

If by $Z(P)$ we denote the set of all connections with zero curvature in P, and by $\Omega(X, \mathfrak{G})$ the set of classes of conjugate homomorphisms $\pi_1(X) \to \mathfrak{G}$, then there arises a mapping

$$\mu : \quad Z(P) \to \Omega(X, \mathfrak{G}),$$

which we must study.

Consider the case in which the fibration P is trivial. Then a connection is given by a differential 1-form α on X with values in the Lie algebra of the group $\mathfrak{G}$. The determination of paths horizontal in the sense of the connection α is carried out by means of the differential equation

$$f^{-1} df = \alpha, \tag{1}$$

where $f : X \to \mathfrak{G}$ is an unknown smooth function. The condition for the triviality of the curvature is expressed by the equality

$$d\alpha + \frac{1}{2}[\alpha, \alpha] = 0. \tag{2}$$

This equality is precisely the condition for complete integrability of equation (1). The holonomy group of the connection α in this case is called the monodromy group of equation (1). The corresponding homomorphism $\omega : \pi_1(X) \to \mathfrak{G}$ will be called the **monodromy homomorphism**.

Let now $\mathfrak{G} = R$, the additive group of real numbers. Then α is an ordinary 1-form on X. Condition (2) means that $d\alpha = 0$. It is easy to see that for $l \in \pi_1(X)$ we have $\omega(l) = \int_l \alpha$. According to de Rham' s theorem, two closed forms have the same periods if and only if they are cohomologous. Further, there always exists a closed form with arbitrary periods. Thus, in this case de Rham' s theorem gives a solution of problems (1) and (2). Therefore the theorem proved below, Theorem 1, may be regarded as a natural generalization of de Rham' s theorem.

2. Let $\{U_i\}$ be a coordinate covering of the manifold X for the bundle P, and let P be determined by transition functions $p_{ij} : U_i \cap U_j \to \mathfrak{G}$. Denote by $\operatorname{Int} P$ the bundle with base X and fiber $\mathfrak{G}$ which, in the covering $\{U_i\}$, is determined by the transition functions $A_{p_{ij}}$, where A_g is the automorphism of the group $\mathfrak{G}$ acting by the formula $A_g(x) = gxg^{-1}$. Then $\operatorname{Int} P$ is a group bundle. Therefore the set $D(\operatorname{Int} P)$ of all its smooth sections is a group. This group acts on the bundle P by means of "left shifts" $L_f (f \in D(\operatorname{Int} P))$. It can be shown that in this way one obtains the group of all automorphisms of the principal bundle P that carry each fiber into itself. Hence there arises a group of transformations L_f^* of the set of all infinitesimal connections in the bundle P, and the set $Z(P)$ is carried into itself. Denote by $H(P)$ the set of orbits of the group $D(\operatorname{Int} P)$ in $Z(P)$. It is easy to see that the holonomy homomorphisms corresponding to two connections lying in the same orbit are conjugate to one another. Therefore a mapping arises

$$\bar{\mu}: \ H(P) \to \Omega(X, \mathfrak{G}).$$

Let us also note that every homomorphism $\pi_1(X) \to \mathfrak{G}$ induces a principal bundle with base X and group $\mathfrak{G}$, having locally constant transition functions (a locally constant fiber bundle). More precisely, there is a one-to-one mapping q from the set $\Omega(X, \mathfrak{G})$ onto $H^1(X, \mathfrak{G})$ (4).

Theorem 1. *The mapping $\bar{\mu}$ is one-to-one, and its image consists of all classes of homomorphisms $\pi_1(X) \to \mathfrak{G}$ for which the corresponding fiber bundles are differentiably equivalent to the bundle P.*

Proof. First of all, it is not difficult to show that if the bundle P has an infinitesimal connection without curvature, then its transition functions p_{ij} can be chosen locally constant. A connection in P is given by a system of 1-forms α_i on U_i with values in G, satisfying on $U_i \cap U_j$ the relation

$$\alpha_j = \mathrm{Ad}(p_{ij}^{-1})\alpha_i + p_{ij}^{-1} dp_{ij}. \tag{3}$$

Consider the bundle $\mathrm{Ad}\, P$ with base X, fiber G, and transition functions $\mathrm{Ad}\, p_{ij}$. Since $dp_{ij} = 0$, it follows from (3) that connections in the bundle P are nothing but 1-forms on X with values in the vector bundle $\mathrm{Ad}\, P$. We denote the space of all such forms by $\mathcal{L}^1(\mathrm{Ad}\, P)$. The condition that the curvature be zero is expressed for $\alpha \in \mathcal{L}^1(\mathrm{Ad}\, P)$ by the formula

$$d\alpha + \frac{1}{2}[\alpha, \alpha] = 0$$

(the commutation has meaning, since $\mathrm{Ad}\, P$ is a bundle of Lie algebras).

Denote the set of all such forms α again by $Z(P)$. One can verify that, when connections are identified with forms from $\mathcal{L}^1(\mathrm{Ad}\, P)$, the operators L_f^* ($f \in D(\mathrm{Int}\, P)$) act according to the formula

$$L_f^* \alpha = \mathrm{Ad}(f^{-1})\alpha + f^{-1} df.$$

It is therefore clear that L_f^* coincides with the operator $C(f^{-1})$, defined in [1]. By Theorem 3 of that paper, we have a one-to-one mapping $p : H(P) \to H^1(X, \mathfrak{G})$, whose image consists of all locally constant bundles differentiably equivalent to the bundle P. Consider the diagram

$$\begin{tikzcd}
H(P) \arrow[rr, "\bar{\mu}"] \arrow[dr, "p"'] & & \Omega(X, \mathfrak{G}) \arrow[dl, "q"] \\
& H^1(X, \mathfrak{G}) &
\end{tikzcd}$$

It is verified that it is commutative. Theorem 1 follows immediately from this.

3. Let us consider the case in which the bundle P is trivial. It follows from Theorem 1 that a homomorphism $\omega : \pi_1(X) \to \mathfrak{G}$ is a monodromy homomorphism of an equation of the form (1) if and only if the corresponding bundle P_ω under the mapping q is differentiably trivial. To decide the question of the triviality of the bundle P_ω, one may use obstruction theory. Suppose that the group $\mathfrak{G}$ is connected, and denote by $\widetilde{\mathfrak{G}}$ its universal covering group. If $\mathfrak{G}$ is not simply connected, then the first obstruction k_ω, arising in the construction of a section in P_ω, lies in the group $H^2(X, \pi_1(\mathfrak{G}))$. It is known that $k_\omega = 0$ if and only if the group of the bundle P_ω can be "lifted" to $\widetilde{\mathfrak{G}}$ [5]. In our case this is equivalent to representability of the homomorphism ω in the form $\omega = \pi\tilde{\omega}$, where $\pi : \widetilde{\mathfrak{G}} \to \mathfrak{G}$ is the covering, and $\tilde{\omega} : \pi_1(X) \to \widetilde{\mathfrak{G}}$ is some homomorphism. Hence the following assertion is obtained.

Theorem 2. Let $\mathfrak{G}$ be a connected Lie group. In order that a homomorphism $\omega : \pi_1(X) \to \mathfrak{G}$ be a monodromy homomorphism for an equation of the form (1), it is necessary that it be representable in the form $\omega = \pi\tilde{\omega}$, where $\tilde{\omega} : \pi_1(X) \to \widetilde{\mathfrak{G}}$ is some homomorphism. In the following particular cases this condition is sufficient:

a) $\pi_k(\mathfrak{G}) = 0$ $(k > 1)$ (i.e. the maximal compact subgroup in $\mathfrak{G}$ is abelian);

b) $H^k(X, \mathbb{Z}) = 0$ $(k \geq 4)$.

We give examples of results following from this theorem.

Corollary 1. If $\pi_1(X)$ is a free group with a finite or countable number of generators and $\mathfrak{G}$ is a connected Lie group, then every homomorphism $\pi_1(X) \to \mathfrak{G}$ is a monodromy homomorphism.

Corollary 2. If $\pi_1(X)$ is a free abelian group with a finite number of generators and $\mathfrak{G} = GL(n, \mathbb{C})$, $SL(n, \mathbb{C})$, or $Sp(n, \mathbb{C})$, then every homomorphism $\pi_1(X) \to \mathfrak{G}$ is a monodromy homomorphism.

On the other hand, for $\mathfrak{G} = GL(n, \mathbb{R})$ it is easy to give an example of a homomorphism that is not a monodromy homomorphism.

4. Suppose now that X is a complex manifold, $\mathfrak{G}$ a complex Lie group, and P a holomorphic principal bundle with base X and group $\mathfrak{G}$. If one considers holomorphic infinitesimal connections with zero curvature in the bundle P, then one obtains a picture analogous to that considered in §§ 1 and 2. In particular, if by $Z_a(P)$ one denotes the set of all holomorphic connections with zero curvature, then the group of all holomorphic sections of the bundle $\operatorname{Int} P$ acts on $Z_a(P)$. Let $H_a(P)$ be the set of orbits of this group in $Z_a(P)$. Then there arises a mapping

$$\mu_a : \ H_a(P) \to \Omega(X, \mathfrak{G}).$$

Similarly to Theorem 1, one proves

Theorem 3. *The mapping μ_a is one-to-one, and its image consists of all classes of homomorphisms $\pi_1(X) \to \mathfrak{G}$ for which the corresponding fiber spaces are analytically equivalent to P.*

Let the fiber space P be trivial. Then the image of μ_a consists of the classes of homomorphisms of monodromy of equations of the form (1), where α is a holomorphic 1-form with values in G. In the case $\dim_C X = 1$, the question considered by us is closely connected with the classical Riemann-Hilbert problem. In this case the assertion of Theorem 3 on the image μ_a was proved by Röhrl (2).

Suppose now that X is a Stein manifold. Then holomorphic fiber spaces with base X and group $\mathfrak{G}$ are analytically equivalent if and only if they are differentiably equivalent (6). Therefore, in Theorem 3 in this case it is enough to consider homomorphisms that generate fiber spaces differentiably equivalent to P. In particular, we have $H_a(P) = H(P)$. If P is trivial, then one may apply the results formulated in § 3.

Moscow State University
named after M. V. Lomonosov

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Note: Figure translations are in progress. See original paper for figures.

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