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Abstract

Full Text

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ON A NEW APPLICATION OF THE FIXED-POINT PRINCIPLE IN THE THEORY OF OPERATORS IN A SPACE WITH AN INDEFINITE METRIC

(Presented by Academician L. S. Pontryagin on 21 X 1963)

1. Let $\mathfrak{H}$ be a Hilbert space in which, along with the usual scalar product (x, y), an indefinite scalar product is given:

$$[x, y] = (Jx, y), \qquad J = P_+ - P_-,$$

where $P_\pm$ are two mutually complementary orthogonal projectors. Put

$$\mathfrak{P}_+ = \{x : \ x \in \mathfrak{H}, \ [x, x] \geq 0\}.$$

A subspace L ($\subset \mathfrak{H}$) is called J-nonnegative if $L \subset \mathfrak{P}_+$. Denote by $\mathfrak{M}_+$ the set of all maximal J-nonnegative subspaces. Denote by $\mathfrak{K}_+$ the set of all linear operators K defined on $\mathfrak{H}_+ = P_+\mathfrak{H}$ and mapping $\mathfrak{H}_+$ into $\mathfrak{H}_- = P_-\mathfrak{H}$ in a nonexpanding manner: $\|K\| \leq 1$.

Analogously one defines J-nonpositive subspaces, the sets $\mathfrak{P}_-$, $\mathfrak{M}_-$, and $\mathfrak{K}_-$.

In ([1], [2]) (see also ([3])) it is shown that the sets $\mathfrak{M}_+$ and $\mathfrak{K}_+$ are in one-to-one correspondence $L \leftrightarrow K$, by virtue of which

$$L = \{x_+ + Kx_+ : \ x_+ \in \mathfrak{H}_+\}$$

(the operator $K \in \mathfrak{K}_+$ is called the angular operator of the subspace $L \in \mathfrak{M}_+$). Of course, the sets $\mathfrak{M}_-$ and $\mathfrak{K}_-$ are in an analogous correspondence.

Denote by $\mathfrak{R}$ the ring of all linear bounded operators A acting in $\mathfrak{H}$, and by $\mathfrak{S}_\infty$ its ideal consisting of all completely continuous $A \in \mathfrak{R}$.

Theorem 1. *Suppose that, for some $A \in \mathfrak{R}$, the following conditions are fulfilled: 1) $0 \neq Ax \in \mathfrak{P}_+$ for $0 \neq x \in \mathfrak{P}_+$; 2) $P_+AP_- \in \mathfrak{S}_\infty$; and 3) there exists at least one $L_0 \in \mathfrak{M}_+$ such that $AL_0 \in \mathfrak{M}_+$.*

Then the operator A maps every $L \in \mathfrak{M}_+$ onto $AL \in \mathfrak{M}_+$, and there exists at least one $L_a \in \mathfrak{M}_+$ such that $AL_a = L_a$.

Proof. Let $L \in \mathfrak{M}_+$ and let K be its angular operator. Then for

$$x = x_+ + Kx_+ \in L \qquad (x_+ \in \mathfrak{H}_+)$$

we shall have

$$P_+Ax = (A_{11} + A_{12}K)x_+, \qquad P_-Ax = (A_{21} + A_{22}K)x_+, \tag{1}$$

where A_{jk} is the operator acting from $\mathfrak{H}_k$ into $\mathfrak{H}_j$ according to the formula

$$A_{jk}x = P_{jAP_{kx}} = P_{jAx}, \quad x \in \mathfrak{H}_k \quad (j, k = 1, 2),$$

with $P_1 = P_+$, $P_2 = P_-$, $\mathfrak{H}_1 = \mathfrak{H}_+$, $\mathfrak{H}_2 = \mathfrak{H}_-$.

Since from $x \in L$ it follows that $Ax \in \mathfrak{P}_+$, i.e.

$$|P_+Ax| \geq |P_-Ax| \qquad (|x| = (x, x)^{1/2}),$$

it follows from $x \in L$, $P_+Ax = 0$ that $Ax = 0$, $x = 0$. Therefore, according to (1),

$$(A_{11} + A_{12}K)x_+ \neq 0$$

for $x_+ \neq 0$, $x_+ \in \mathfrak{H}_+$, whatever the operator $K \in \mathfrak{K}_+$ may be. In particular, the operator

$$W_0 = A_{11} + A_{12}K_0,$$

acting in $\mathfrak{H}_+$, where K_0 is the angular operator for L_0, vanishes only at zero. On the other hand, by the condition

$$L_0' = AL_0 \in \mathfrak{M}_+,$$

and therefore

$$P_+L_0' = \mathfrak{H}_+,$$

i.e.

$$\mathfrak{H}_+ = \{P_+Ax : \ x \in L_0\} = \{(A_{11} + A_{12}K_0)x_+ : \ x_+ \in \mathfrak{H}_+\}.$$

Consequently, the operator

$$W_0 = A_{11} + A_{12}K_0$$

maps $\mathfrak{H}_+$ one-to-one and continuously onto itself and hence, by Banach' s theorem, has a continuous inverse.

Comparing the operator

$$W = A_{11} + A_{12}K$$

for an arbitrary $K \in \mathfrak{K}_+$ with the operator W_0, we observe that their difference

$$W - W_0 = A_{12}(K - K_0) \in \mathfrak{S}_\infty$$

(by the condition $A_{12} = P_+AP_- \in \mathfrak{S}_\infty$), and consequently

$$W = W_0(I + T),$$

where

$$T = W_0^{-1}(W - W_0) \in \mathfrak{S}_\infty.$$

Since W vanishes only at zero, we conclude:

a) For any $K \in \mathfrak{K}_+$, the operator $W(K) = A_{11} + A_{12}K$ maps $\mathfrak{H}_+$ onto itself one-to-one and continuously.

Thus, for any $K \in \mathfrak{K}_+$, the fractional-linear transformation

$$K' = (A_{21} + A_{22}K)(A_{11} + A_{12}K)^{-1} \ (= \Phi(K)). \tag{2}$$

is meaningful.

According to (1), for $L(\in \mathfrak{M}_+)$ with angular operator K we have: $Ax = P_+Ax + K'P_+Ax$ $(x \in L)$. Since, as was proved, P_+Ax runs through all of $\mathfrak{H}_+$ when x runs through L, for $L' = AL(\subset \mathfrak{P}_+)$ we obtain: $L' = \{x_+ + K'x_+ : x_+ \in \mathfrak{H}_+\}$. Hence the subspace $L' \in \mathfrak{M}_+$, and K' is its angular operator. The first assertion is proved.

At the same time it has been shown that $\Phi(K)$ maps $\mathfrak{K}_+$ into itself. As is known (4), $\mathfrak{K}_+$ is a convex bicompact in the weak topology. If we show that the mapping $\Phi(K)$ is continuous in this topology, then, by the Schauder–Tikhonov theorem, $\Phi(K)$ will have a fixed point, and thereby the second assertion of the theorem will be proved.

From conclusion a) it follows that the operator $A_{11} = W(0)$ is continuously invertible. Therefore $W(K) = A_{11}(I - SK)$, where $S = -A_{11}^{-1}A_{12}$ is an operator mapping $\mathfrak{H}_-$ completely continuously into $\mathfrak{H}_+$. Put $s = \|S\|$, and let $\psi(\in \mathfrak{H}_-)$ be a unit vector ($\|\psi\| = 1$) for which $S\psi(\in \mathfrak{H}_+)$ has norm equal to s. Then $S\psi = s\varphi$, $\|\varphi\| = 1$. Obviously, for any q $(0 \leq q \leq 1)$ the operator $K_q = q(\cdot, \varphi)\psi \in \mathfrak{K}_+$, and for it $(I - SK_q)\varphi = (1 - qs)\varphi$. Since the number $1 - qs$ $(0 \leq q \leq 1)$ must be nonzero, $s = \|S\| < 1$. The function $\Phi(K)$ $(K \in \mathfrak{K}_+)$ can be written in the form

$$\Phi(K) = (A_{21} + A_{22}K)(I - SK)^{-1}A_{11}^{-1}.$$

Since, as was proved, $\|S\| < 1$, on $\mathfrak{K}_+$ the function $\Phi(K)$ is the uniform limit, in the uniform norm, of the sequence of functions

$$\Phi_n(K) = (A_{21} + A_{22}K)\sum_{p=0}^{n}(SK)^p A_{11}^{-1}.$$

Taking into account that the operator S, being completely continuous, can be approximated with arbitrary accuracy in the uniform norm by a finite-dimensional operator, it is not difficult to show that each function $\Phi_n(K)$ is continuous on the bicompact $\mathfrak{K}_+$ in the weak topology. Consequently, the function $\Phi(K)$ has the same property, and the theorem is completely proved.

Theorem 2. Let, for the operator $A(\in \mathfrak{R})$, the following conditions be satisfied: 1) the operator is continuously invertible: $A^{-1} \in \mathfrak{R}$; 2) $A\mathfrak{P}_+ \subset \mathfrak{P}_+$; and 3) the operators P_+AP_- and P_-AP_+ are completely continuous. Then there exist at least one $L_+ \in \mathfrak{M}_+$ and one $L_- \in \mathfrak{M}_-$ such that $AL_+ = L_+$ and $AL_- = L_-$.

Proof. If $A\mathfrak{P}_+ \subset \mathfrak{P}_+$ ($A\mathfrak{P}_- \subset \mathfrak{P}_-$) and there exists $A^{-1}(\in \mathfrak{R})$, then, obviously, $A^{-1}\mathfrak{P}_- \subset \mathfrak{P}_-$ ($A^{-1}\mathfrak{P}_+ \subset \mathfrak{P}_+$). Therefore, when conditions 1) and 2) of Theorem 2 are fulfilled, we always have $A^{-1}\mathfrak{P}_\pm \subset \mathfrak{P}_\pm$, and A maps any $L \in \mathfrak{M}_+$ ($L \in \mathfrak{M}_-$) into $AL \in \mathfrak{M}_+$ ($AL \in \mathfrak{M}_-$). Thus, Theorem 2 is a simple consequence of Theorem 1.

2. An operator $U(\in \mathfrak{R})$ is called ***J*-unitary** if $U\mathfrak{H} = \mathfrak{H}$ and $U^*JU = J$. Denote by $\mathfrak{G}$ the set of all J-unitary operators U for which the condition $P_+UP_- \in \mathfrak{S}_\infty$ is fulfilled. It is not difficult to show that this condition for a J-unitary operator U is equivalent to the condition $P_-UP_+ \in \mathfrak{S}_\infty$, and also that:

1°. The set $\mathfrak{G}$ is a subgroup of the group of all J-unitary operators in $\mathfrak{H}$, and $\mathfrak{G} = \mathfrak{G}^*$. The latter equality means that if $U \in \mathfrak{G}$, then also $U^* \in \mathfrak{G}$.

We shall show that, for each $U \in \mathfrak{G}$, one can construct a unitary operator V ($V^*V = VV^* = I$) such that $U - V \in \mathfrak{S}_\infty$.

Indeed, for $U \in \mathfrak{G}$ we have $T = U - JUJ = (P_+ + P_-)U(P_+ + P_-) - (P_+ - P_-)U(P_+ - P_-) = 2(P_+UP_- + P_-UP_+) \in \mathfrak{S}_\infty$, since $(U^*)^{-1} = JUJ = U - T$ and $U^*U = I + S$, where $S = U^*T \in \mathfrak{S}_\infty$. Define the self-adjoint-

operator $S_1 \in \mathfrak{S}_\infty$ by the equality $S_1 = (I + S)^{1/2} - I$. Then $I + S = (I + S_1)^2$, and the continuously invertible operator $V = U(I + S_1)^{-1}$ will be unitary, since $V^*V = (I + S_1)^{-1}(I + S)(I + S_1)^{-1} = I$. On the other hand, $VS_1 \in \mathfrak{S}_\infty$.

On the basis of the general theorem of perturbation theory (see, for example, (^{5}), Theorem 2.3) it follows from this that:

2°. Every nonunitary point ρ ($|\rho| \neq 1$) of the spectrum $\sigma(U)$ of the operator $U \in \mathfrak{G}$ is an eigenvalue of this operator, to which there corresponds a finite-dimensional normally separated root subspace $\mathfrak{L}_\rho(U)$.

Consequently, the nonunitary spectrum

$$\sigma_0(U) = \{\rho : \rho \in \sigma(U),\ |\rho| \neq 1\}$$

of the operator $U \in \mathfrak{G}$ consists of isolated points. Let us note that the nonunitary spectrum $\sigma_0(U)$ of any J-unitary operator U has the following properties: if $\rho \in \sigma_0(U)$, then also $\bar{\rho}^{-1} \in \sigma_0(U)$, and if the root subspace $\mathfrak{L}_\rho(U)$ is finite-dimensional and normally separated, then the root subspace $\mathfrak{L}_{1/\bar{\rho}}(U)$ has the same properties; moreover, in $\mathfrak{L}_\rho$ and in $\mathfrak{L}_{1/\bar{\rho}}$ the operator U has identical elementary divisors (with ρ replaced by $1/\bar{\rho}$); furthermore, in this case $\mathfrak{L}_\rho$ and $\mathfrak{L}_{1/\bar{\rho}}$ are skewly related,* and the subspace $\mathfrak{N}$ of all vectors J-orthogonal to the direct sum $\mathfrak{L}_\rho + \mathfrak{L}_{1/\bar{\rho}}$ is invariant with respect to U, and

$$(U - \rho I)\mathfrak{N} = (U - \bar{\rho}^{-1}I)\mathfrak{N} = \mathfrak{N}$$

(cf. (^{6},^{7})).

Theorem 3. Suppose that the nonunitary spectrum σ_0 of the operator $U \in \mathfrak{G}$ is in some way divided into two disjoint sets σ_I and $\sigma_0 \setminus \sigma_I$, symmetric with respect to the unit circle. Then there exist two subspaces $L_\pm \in \mathfrak{M}_\pm$ with the properties: 1) $UL_\pm = L_\pm$ and 2) the nonunitary spectrum of the restriction of the operator U to $L_\pm$ coincides with σ_I.

From properties 1) and 2), taking into account that $L_\pm \in \mathfrak{M}_\pm$, it follows (cf. Lemma II.9 in ([7])): 3) if $\rho \in \sigma_I$, then $\mathfrak{L}_\rho(U) \subset L_\pm$ and $\mathfrak{L}_{1/\bar\rho} \cap L_\pm = \{0\}$.

Proof. According to Theorem 2, there exists $L_+^{(0)} \in \mathfrak{M}_+$ with the first property: $UL_+^{(0)} = L_+^{(0)}$. If $L_+^{(0)}$ does not possess the second property, then it may be modified into some $L_+ \in \mathfrak{M}_+$ with properties 1) and 2) by the method which was applied in the analogous case in ([8]). The existence of $L_- \in \mathfrak{M}_-$ with properties 1) and 2) is obtained from "symmetric" considerations.

3. A linear operator H, acting in $\mathfrak{H}$ with dense domain of definition $\mathfrak{D}(H)$, is called J-self-adjoint if JH is an ordinary self-adjoint operator in $\mathfrak{H}$ (for other equivalent definitions see ([6,7])). Such an operator is, evidently, closed. Therefore, if $\mathfrak{H}_+ \subset \mathfrak{D}(H)$, then H will be a bounded operator on $\mathfrak{H}_+$, or, what is the same thing, the operators $H_{11} = P_+HP_+$ and $H_{21} = P_-HP_+$ will be bounded. In this case the closure of the operator $H_{12} = P_+HP_-$ will also be bounded, since $H_{12}^* = -H_{21}$. As for the operator $H_{22} = P_-HP_-$, it is easy to see that, like the operator H_{11}, it will be a self-adjoint operator, but unbounded, unless $\mathfrak{D}(H) = \mathfrak{H}$.

Theorem 4. Let H be a J-self-adjoint operator such that: a) $\mathfrak{H}_+ \subset \mathfrak{D}(H)$ and b) the operator $H_{12}(H_{22} - \xi I)^{(-1)}$, for some** $\xi \notin \sigma(H_{22})$, maps $\mathfrak{H}$ completely continuously into $\mathfrak{H}_+$. Then the nonreal spectrum $\sigma_0(H)$ of the operator H consists of isolated eigenvalues with normally separated root subspaces. Moreover, if this spectrum is in some way divided into two disjoint sets Λ and $\bar\Lambda = \sigma_0 \setminus \Lambda$, symmetric with respect to the real axis, then there is always a subspace $L_+ \in \mathfrak{M}_+$ with the properties: 1) $L_+ \subset \mathfrak{D}(H)$, 2) $HL_+ \subset L_+$, and 3) the spectrum of the restriction of H to L_+ coincides with Λ.

Proof. By elementary calculations it is shown that, for $\operatorname{Im}\xi$ sufficiently large in absolute value, the operator H (with properties a) and b)) there exists a J-unitary Cayley transform

$$U_\zeta = (H - \bar\zeta I)(H - \zeta I)^{-1},$$

where

$$\begin{aligned} P_+U_\zeta P_+ &= \left[I + (\zeta - \bar\zeta)(I - T)^{-1}(H_{11} - \zeta I)^{-1}\right] P_+, \\ P_+U_\zeta P_- &= -(\zeta - \bar\zeta)(I - T)^{-1}(H_{11} - \zeta I)^{-1} H_{12}(H_{22} - \zeta I)^{-1} P_-, \end{aligned} \tag{3}$$

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* That is, in neither of the subspaces $\mathfrak{L}_\rho$ and $\mathfrak{L}_{1/\bar\rho}$ is there a vector ($\neq 0$) J-orthogonal to the other subspace.

** If condition b) is satisfied for some $\xi \in \sigma(H_{22})$, then it is satisfied for all $\xi \in \sigma(H_{22})$; by $(H_{22} - \xi I)^{(-1)}$ is denoted the resolvent at the point ξ of the restriction of the operator H_{22} to $\mathfrak{H}_-$.

where

$$T = (H_{11} - \zeta I)^{-1} H_{12} (H_{22} - \zeta I)^{-1} H_{21} \in \mathfrak{S}_\infty, \quad \|T\| < 1.$$

From the second relation (3) it follows that $U_\zeta \in \mathfrak{S}$, whence we obtain the first assertion of the theorem. According to Theorem 3, there will exist an $L_+ \in \mathfrak{M}_+$ such that $U_\zeta L_+ = L_+$, and the spectrum of the restriction of U_ζ to L_+ coincides with the image Λ under the mapping

$$\mu = (\lambda - \bar{\zeta})(\lambda - \zeta)^{-1}.$$

Using the boundedness of H_{11} and the first relation (3), one can show that $(U_\zeta - I)L_+ = L_+$. The three indicated properties of L_+ with respect to U_ζ are equivalent to the properties 1), 2), and 3) asserted in Theorem 4.

Let us note that condition b) will be satisfied if it is replaced by the condition

$$P\_HP_+ \in \mathfrak{S}_\infty;$$

in this case Theorem 4 becomes a theorem proved by another method by G. Langer (^{7},^{8}) (under the assumption of separability of $\mathfrak{H}$). In turn, G. Langer' s theorem was an important generalization of a well-known theorem of L. S. Pontryagin (^{9}) (see also (^{10})), into which Theorem 4 passes when

$$\varkappa = \dim \mathfrak{H}_+ < \infty$$

(in this case conditions a) and b) of Theorem 4 are automatically satisfied).

The method of proof of L. S. Pontryagin' s theorem and of its generalizations on the basis of the fixed-point principle was first proposed by the author in (^{11}) (see also (^{10})). This method received further development in a witty note by M. L. Brodskii (^{12}), in which, in particular, for the case $\varkappa < \infty$, a proposition more complete than Theorem 1 was proved. As in (^{12}), Theorem 1 can be reformulated and proved for operators acting in a Banach space endowed with an indefinite metric.

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