ON THE SOLVABILITY OF FILTRATION PROBLEMS WITH FREE BOUNDARIES

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Abstract

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In the present work, for proving theorems on the existence of solutions of problems of linear and nonlinear filtration, a new method is proposed, by means of which it is possible to establish more general theorems than those established by other methods ([1–3]). This method can also serve for the construction of solutions.

The motion of a fluid in an earth dam with vertical walls on an impermeable base is considered. The filtration region in the z-plane is bounded by the following lines: the vertical walls M_1M_2 and M_4M_5 with equations $x = x_1,\ x = x_2$; the impermeable base M_1M_5 with equation $y = 0$; the seepage face M_3M_4 with equation $x = ay + x_0$, and the unknown free surface (depression curve) M_2M_3. As is known ([1]), in the case when Darcy' s law is fulfilled, the complex filtration potential $w(z) = \varphi + i\psi$ is an analytic function and on the boundaries of the filtration region satisfies the following boundary conditions:

$$\psi = 0, \quad y = 0 \quad \text{on } M_1M_5; \tag{1}$$

$$\varphi = \varphi_i, \quad x = x_i \quad \text{on } M_1M_2\ (i = 1) \text{ and } M_4M_5\ (i = 2); \tag{2}$$

$$\varphi = -ky + \varphi_0, \quad \psi = \varepsilon x + \psi_0 \quad \text{on } M_2M_3; \tag{3}$$

$$\varphi = -ky + \varphi_0, \quad x = ay + x_0 \quad \text{on } M_3M_4, \tag{4}$$

where k is the filtration coefficient; ε is the infiltration coefficient ($\varepsilon > 0$) or evaporation coefficient ($\varepsilon < 0$).

In the case when a problem with an inclined seepage face is considered, it is natural to assume that on it, as on the free surface, infiltration or evaporation may occur. Therefore, instead of the condition (4) usually adopted on the seepage face, one may require that the condition

$$\psi = \varepsilon x + \psi_0, \quad x = ay + x_0 \quad \text{on } M_3M_4 \tag{5}$$

be fulfilled.

Thus, it is required to determine the form of the depression curve M_2M_3 and, in the domain bounded by this curve and the polygon $M_2M_1M_5M_4M_3$, to find an analytic function $w = \varphi + i\psi$ satisfying the boundary conditions (1)–(4) or (1)–(3), (5). We shall solve the boundary-value problem (1)–(4) and, for definiteness, fix some of the constants entering into the boundary conditions by prescribing the ordinate y_3 of the point M_3 and the values $\varphi_0, \varphi_1, \varphi_2, x_2$, with $\varphi_1 < -ky_3 + \varphi_0 < \varphi_2$. The assignment of these constants uniquely determines the abscissa x_3 of the point M_3 and the constant x_0; the constants x_1 and ψ_0 will be found in the process of solution. By the conditions (1)–(4), in the plane of the variables φ, y there is determined the quadrilateral $L_\varphi = M_1'M_2'M_4'M_5'$, while on the side $M_2'M_4'$ the point M_3' is fixed. The functions $x = x(\varphi, y)$ and $\psi = \psi(\varphi, y)$ satisfy

system

$$\psi_\varphi = x_y; \qquad \psi_y = -\frac{1 + x_y^2}{x_\varphi}, \tag{6}$$

obtained by transforming the Cauchy–Riemann system, with

$$\left|\frac{dw}{dz}\right|^2 = \frac{1 + x_y^2}{x_\varphi^2}.$$

Eliminating the function ψ from system (6) and denoting $x_\varphi \equiv R,\ x_y \equiv \tau$, we express φ and y as functions of the variables τ, R. In this way we obtain the linear elliptic system

$$\varphi_\tau = y_R; \qquad \varphi_R + 2\frac{\tau}{R}y_R + \frac{1 + \tau^2}{R^2}y_\tau = 0. \tag{7}$$

From the boundary conditions (1)–(4) we find the images of the sides of the polygon L_φ in the plane of the variables τ, R. We obtain

$$\begin{aligned} &\tau = 0 && \text{on} \quad M_2'M_1'M_5'M_4', \\ &\tau = kR + a && \text{on} \quad M_3'M_4', \\ &\tau(\varepsilon + k) - k\varepsilon R - \frac{1 + \tau^2}{R} = 0 && \text{on} \quad M_2'M_3'. \end{aligned} \tag{8}$$

Elementary calculations show that, except for the case $\varepsilon > k$, in the plane of the variables τ, R there always exists a domain $D(L_R)$, bounded by the lines with equations (8), and containing no points of the line $R = 0$. The latter ensures the boundedness of $|dw/dz| = \sqrt{1+\tau^2}/R$. Let the points of the boundary L_R corresponding to the points M_i' be denoted by M_i''.

Construct a quasiconformal mapping $\varphi = M(\tau, R),\ y = N(\tau, R)$ of the domain $D(L_R)$ onto the interior of the quadrilateral L_φ as a solution of the linear elliptic system (7), with the points M_i'' corresponding to the points M_i' $(i = 2, 3, 4)$. The existence of such a mapping follows from the generalized Riemann theorem for quasiconformal mappings ([4]). Note that, in order to construct a quasiconformal mapping of $D(L_R)$ onto $D(L_\varphi)$, it is necessary, for the linear elliptic system (7), to solve a boundary-value problem with boundary conditions linear in φ and y on the boundary L_R; moreover, by means of a series of transformations of the independent variables and functions this problem is reduced to a linear Hilbert problem for an analytic function. It is easy to verify by direct calculations that $D(\varphi, y)/D(\tau, R) > 0$ inside the domain $D(L_R)$ and can vanish or become infinite only at the points M_2'' and M_4''. Consequently, there exists an inverse mapping $\tau = f_1(\varphi, y),\ R = f_2(\varphi, y)$. Thus we have $x_\varphi = f_2(\varphi, y),\ x_y = f_1(\varphi, y)$, whence

$$x = \int_{(\varphi_2, 0)}^{(\varphi, y)} f_2(\varphi, y)\, d\varphi + f_1(\varphi, y)\, dy + x_2 \equiv F_1(\varphi, y); \tag{9}$$

$$\psi = \int_{(\varphi_2, 0)}^{(\varphi, y)} f_1(\varphi, y)\, d\varphi + \frac{1 + [f_1(\varphi, y)]^2}{f_2(\varphi, y)}\, dy \equiv F_2(\varphi, y). \tag{10}$$

By construction, the functions $x = F_1(\varphi, y)$ and $\psi = F_2(\varphi, y)$ satisfy the boundary conditions (1)–(4). The curvilinear integrals in formulas (9), (10) do not depend on the path of integration by virtue of equations (6). Since, by construction, $x_\varphi = R \neq 0$ in the domain $D(L_\varphi)$, there exists an implicit function $\varphi = G_1(x, y)$ satisfying the equation $x = F_1(\varphi, y)$. Then from formula (10) we find $\psi = F_2[G_1(x, y), y] \equiv G_2(x, y)$. The function thus constructed, $w = \varphi + i\psi$, is analytic and satisfies the boundary conditions (1)–(4). The equation of the contour L_z bounding the filtration domain can be obtained from the formulas

$$x = F_1[M(\tau, R), N(\tau, R)], \qquad y = N(\tau, R), \tag{11}$$

when $(\tau, R) \in L_R$. Here the mapping defined by equalities (11) satisfies, in the domain $D(L_R)$, the linear elliptic system of equations

$$x_\tau = \tau y_\tau + R y_R; \qquad -x_R = \frac{1+\tau^2}{R}\, y_\tau + \tau y_R, \tag{12}$$

i.e., it is quasiconformal. Hence, in particular, the closedness of the constructed contour L_z follows.

By the same method one establishes the solvability of the filtration problem in the case of inclined walls M_1M_2 and M_4M_5.

The solution of the boundary-value problem (1)–(3), (5) is constructed in a completely analogous way; only, as the basis, one takes a known contour in the plane of the variables x, ψ, while the role of the "hodograph" plane is played by the plane of the variables $r = y_\psi,\ t = y_x$.

In the case of a nonlinear filtration law, the equations obtained by S. A. Khristianovich ([2]) for the complex filtration potential $w = \varphi + i\psi$ can be written in the form

$$-\psi_x = \rho(Q)\varphi_y; \qquad \psi_y = \rho(Q)\varphi_x, \tag{13}$$

where $Q^2 = \varphi_x^2 + \varphi_y^2,\ \varphi = -kH + C,\ k$ is the coefficient of filtration, and H is the piezometric head.

It is assumed that the ellipticity condition for this system is satisfied,

$$0 < \rho \frac{d}{dQ}(Q\rho) < \infty.$$

On the boundaries of the filtration domain the same conditions (1)–(4) are satisfied as in the linear case; consequently, in the (φ, y)-plane the form of the domain $D(L_\varphi)$ does not change. The system corresponding to system (7) assumes the form

$$\varphi_\tau = y_R; \qquad (1-M)\varphi_R + 2\frac{\tau}{R}y_R + \frac{1+\tau^2}{R^2}y_\tau = 0, \tag{14}$$

where $M = \dfrac{1}{R^2Q}\dfrac{d\rho}{d(Q\rho)}$.

In the boundary conditions (8) only the last condition changes, taking the form

$$\tau(\varepsilon + k\rho) - k\varepsilon R - \frac{1+\tau^2}{R}\rho = 0,$$

and this change will not affect the existence of the corresponding domain $D(L_R)$. Consequently, the original problem has been reduced to the problem of a quasi-conformal mapping of the domain $D(L_R)$ onto the domain $D(L_\varphi)$ by solutions of the linear elliptic system (14).

Thus, from the point of view of the method proposed by us, the distinction between problems with linear and nonlinear filtration laws is essentially erased.

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[4] Z. Ya. Shapiro, *DAN*, **30**, No. 8 (1941).

Note: Figure translations are in progress. See original paper for figures.