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Abstract

Full Text

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FUNDAMENTAL THEOREMS FOR SINGULAR INTEGRAL EQUATIONS WITH SHIFT

(Presented by Academician I. N. Vekua, 16 IX 1963)

1. Let Γ denote the union of a finite number of simple closed Lyapunov contours $\Gamma_1, \dots, \Gamma_p$ ($\Gamma = \Gamma_1 + \dots + \Gamma_p$), having no points in common, and let S^+ be the finite domain of the plane $z = x + iy$ bounded by the contour Γ. We shall assume that the positive direction of Γ leaves S^+ on the left. Let $\alpha(t)$ be a certain function of a point of the contour Γ satisfying the following conditions: 1) $\alpha(t)$ maps the contour Γ homeomorphically onto itself (i.e. each Γ_i is mapped homeomorphically onto itself) with preservation of the direction of traversal *; 2) $\alpha'(t) = d\alpha(t)/dt \neq 0$ everywhere on Γ, and $\alpha'(t) \in H$ (Hölder condition); 3) for some fixed natural number n

$$\alpha_n(t) \equiv \alpha[\alpha_{n-1}(t)] = t \qquad (\alpha_0(t) \equiv t). \tag{1,1}$$

In the present paper we shall prove the fundamental theorems for the singular integral equation with shift of the form

$$T\varphi \equiv \sum_{l=0}^{n-1} \left\{ A_l(t_0)\, \varphi[\alpha_l(t_0)] + \frac{1}{\pi i} \int_\Gamma \frac{K_l(t_0,t)\varphi(t)\, dt}{t - \alpha_l(t_0)} \right\} = f(t_0), \tag{1,2}$$

where $A_l(t_0)$, $K_l(t_0,t)$, and $f(t_0)$ are given functions on Γ satisfying condition H. The unknown function will be sought in the class of functions satisfying condition H.

The investigation of equation (1,2) is carried out by reducing it to a system of singular integral equations. The basic idea of this method may be found in ([1]). Some particular cases of equation (1,2) were studied in the works ([3],[4]).

2. We shall call equation (1,2) of **normal type** if

$$\det[G(t) \pm K(t,t)] \neq 0 \qquad (t \in \Gamma), \tag{2,1}$$

where $G(t_0)$ and $K(t_0,t)$ have the form (4,6) and (4,7).

By the index of the operator T, or of the equation $T\varphi = f$, we shall mean the integer

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$$\varkappa(n) = \operatorname{ind} T = \frac{1}{2\pi}\left\{\arg\frac{\det[G(t) - K(t,t)]}{\det[G(t) + K(t,t)]}\right\}_\Gamma . \tag{2,2}$$

The operator T and the operator T', defined by the formula

$$T'\psi \equiv \sum_{l=0}^{n-1}\left\{\alpha_l'(t_0)A_{n-l}[\alpha_l(t_0)]\,\psi[\alpha_l(t_0)] - \frac{1}{\pi i}\int_\Gamma \frac{K_{n-l}(t,t_0)\psi(t)\,dt}{\alpha_{n-l}(t) - t_0}\right\}, \tag{2,3}$$

where $A_n[\,] = A_0[\,]$, $K_n[\,] = K_0[\,]$, will be called adjoint. Obviously, $(T')' = T$.

Let us note that from the normality of the operator T, generally speaking, there follows the normality of the operator T' adjoint to it. However, below we shall see the expediency of considering the operator T', as well as the above condition of normality and the introduction of the notion of index.

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* Similarly one may consider the case when $\alpha(t)$ changes the direction of traversal of the contour Γ.

In what follows, unless the contrary is stated, we shall always assume that T is an operator of normal type. Everywhere below, linear independence is understood to mean linear independence over the field of complex numbers.

3. The main results of the paper are the following theorems.

Theorem 1. *The number of linearly independent solutions of the equation $T\varphi = 0$ is finite.*

Theorem 2. *The number of linearly independent solutions of the equation $T'\psi = 0$, adjoint to the equation $T\varphi = 0$, is finite.*

Theorem 3. *The difference between the number k of linearly independent solutions of the equation $T\varphi = 0$ and the number k' of linearly independent solutions of the adjoint equation $T'\psi = 0$ is equal to the index of the operator T, i.e.*

$$k - k' = \varkappa(n). \tag{3,1}$$

Theorem 4. *A necessary and sufficient condition for solvability of the equation $T\varphi = f$ is that*

$$\int_\Gamma f(t)\psi^\delta(t)\,dt = 0, \tag{3,2}$$

where $\psi^\delta(t)$ $(\delta = 1, 2, \dots, k')$ is a complete system of linearly independent solutions of the equation $T'\psi = 0$.

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4. We proceed to the proof of Theorems 1-4.

Denoting

$$\varphi[\alpha_l(t)] = \Phi_l(t) \qquad (l = 0, 1, \dots, n-1) \tag{4,1}$$

and observing that $\Phi_l[\alpha_j(t)] = \Phi_{l+j}(t)$ $(j = 0, 1, \dots)$, equation (1,2) can be reduced to the system

$$D(t_0, t)\,\vec{\Phi}(t) = \mathbf{F}(t_0), \tag{4,2}$$

where

$$G(t_0) = \left\| \begin{matrix} A_0(t_0) & A_1(t_0) & \cdots & A_{n-1}(t_0) \\ A_{n-1}[\alpha(t_0)] & A_0[\alpha(t_0)] & \cdots & A_{n-2}[\alpha(t_0)] \\ \cdots & \cdots & \cdots & \cdots \\ A_1[\alpha_{n-1}(t_0)] & A_2[\alpha_{n-1}(t_0)] & \cdots & A_0[\alpha_{n-1}(t_0)] \end{matrix} \right\|, \tag{4,3}$$

$$(t - t_0)^{-1} K(t_0, t) =$$

$$= \left\| \begin{matrix} \dfrac{K_0(t_0, t)}{t - t_0} & \dfrac{K_1[t_0, \alpha(t)]\alpha'(t)}{\alpha(t) - \alpha(t_0)} & \cdots & \dfrac{K_{n-1}[t_0, \alpha_{n-1}(t)]\alpha'_{n-1}(t)}{\alpha_{n-1}(t) - \alpha_{n-1}(t_0)} \\ \dfrac{K_{n-1}[\alpha(t_0), t]}{t - t_0} & \dfrac{K_0[\alpha(t_0), \alpha(t)]\alpha'(t)}{\alpha(t) - \alpha(t_0)} & \cdots & \dfrac{K_{n-2}[\alpha(t_0), \alpha_{n-1}(t)]\alpha'_{n-1}(t)}{\alpha_{n-1}(t) - \alpha_{n-1}(t_0)} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{K_1[\alpha_{n-1}(t_0), t]}{t - t_0} & \dfrac{K_2[\alpha_{n-1}(t_0), \alpha(t)]\alpha'(t)}{\alpha(t) - \alpha(t_0)} & \cdots & \dfrac{K_0[\alpha_{n-1}(t_0), \alpha_{n-1}(t)]\alpha'_{n-1}(t)}{\alpha_{n-1}(t) - \alpha_{n-1}(t_0)} \end{matrix} \right\|, \tag{4,4}$$

$$\vec{\Phi}(t) = \{\Phi_0(t), \dots, \Phi_{n-1}(t)\}, \tag{4,5}$$

and $\mathbf{F}(t_0) = \{f(t_0),\, f[\alpha(t_0)], \dots, f[\alpha_{n-1}(t_0)]\}$. It is easy to see that $G(t_0)$, $K(t_0, t)$, and $\mathbf{F}(t_0)$ satisfy condition H. In what follows, system (4,2) will be considered independently of conditions (4,1).

Theorem 5. *For the solvability of equation (1,2) it is necessary and sufficient that system (4,2) be solvable; moreover, if the function $\varphi(t)$ is a solution of equation (1,2), then the vector $\{\varphi(t),\, \varphi[\alpha(t)], \dots, \varphi[\alpha_{n-1}(t)]\}$ is a solution of system (4,2). Conversely, if the vector (3) is a solution of system (4,2), then the function*

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$$\omega(t) = \frac{1}{n}\{\Phi_0(t) + \Phi_1[\alpha_{n-1}(t)] + \dots + \Phi_{n-1}[\alpha(t)]\} \tag{4,6}$$

is a solution of equation (1,2).

This theorem for an equation of the type $T'\varphi = f$ was proved in the paper ([3]). It should be noted that in the proof of this theorem, as well as of the following Theorem 6, the fulfillment of the normality condition is not assumed.

Theorem 6. *If the homogeneous system of singular integral equations*

$$D(t_0, t)\vec{\Phi}(t) = 0$$

has a finite number $\tilde{k}$ of linearly independent solutions, then this number coincides with the number of linearly independent solutions of the equation $T\varphi = 0$.

Proof. Let the equation $T\varphi = 0$ have k linearly independent solutions $\varphi^\delta(t)$ $(\delta = 1, 2, \dots, k)$. Then, as we already know, the vectors

$$\vec{\Phi}^\delta(t) = \{\varphi^\delta(t), \varphi^\delta[\alpha(t)], \dots, \varphi^\delta[\alpha_{n-1}(t)]\} \quad (\delta = 1, 2, \dots, k)$$

will be solutions of the system $D(t_0, t)\vec{\Phi}(t) = 0$. Consequently,

$$k \leq \tilde{k}. \tag{4,7}$$

Let, further, the vectors

$$\vec{\Phi}_j^\delta(t) = \{\Phi_{0,j}^\delta(t), \Phi_{1,j}^\delta(t), \dots, \Phi_{n-1,j}^\delta(t)\} \quad (\delta = 1, 2, \dots, \tilde{k};\ j = 1, 2) \tag{4,8}$$

be linearly independent solutions of the system $D(t_0, t)\vec{\Phi}(t) = 0$. Then, as was shown above, the vectors

$$\widetilde{\mathbf{W}}_j^\delta(t) = \{\tilde{\omega}_j^\delta(t), \tilde{\omega}_j^\delta[\alpha(t)], \dots, \tilde{\omega}_j^\delta[\alpha_{n-1}(t)]\} \quad (\delta = 1, 2, \dots, \tilde{k},\ j = 1, 2), \tag{4,9}$$

where

$$\tilde{\omega}_j^\delta(t) = \Phi_{0,j}^\delta(t) + \Phi_{n-1,j}^\delta[\alpha(t)] + \dots + \Phi_{1,j}^\delta[\alpha_{n-1}(t)] \quad (\delta = 1, 2, \dots, \tilde{k};\ j = 1, 2) \tag{4,10}$$

will be solutions of the same system and, consequently, they will be linear combinations of the vectors $\vec{\Phi}_j^\delta(t)$ $(\delta = 1, 2, \dots, \tilde{k})$, i.e.

$$\widetilde{\mathbf{W}}_j^\delta(t) = \sum_{\beta=1}^{\tilde{k}} C_{\delta\beta}^{(j)} \vec{\Phi}_1^\beta(t) \quad (j = 1, 2;\ \delta = 1, 2, \dots, \tilde{k}). \tag{4,11}$$

Let $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$ be constants such that

$$\det\left\|C_{\beta\delta}^{(1)} + \lambda C_{\beta\delta}^{(2)}\right\| \neq 0 \quad (\lambda = \lambda_2/\lambda_1).$$

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It is obvious that the first component of the vector

$$\mathbf{W}^{\delta}(t) = \lambda_1 \widetilde{\mathbf{W}}_1^{\delta}(t) + \lambda_2 \widetilde{\mathbf{W}}_2^{\delta}(t) = \{\omega^{\delta}(t), \omega^{\delta}[\alpha(t)], \dots, \omega^{\delta}[\alpha_{n-1}(t)]\}, \qquad (4{,}12)$$

where

$$\omega^{\delta}(t) = \lambda_1 \tilde{\omega}_1^{\delta}(t) + \lambda_2 \tilde{\omega}_2^{\delta}(t) \quad (\delta = 1, 2, \dots, \tilde{k}), \qquad (4{,}13)$$

will be solutions of the equation $T\varphi = 0$. The system of functions (4,13) will be linearly independent. Suppose the contrary; then there exist constants d_δ, not all equal to zero, such that

$$\sum_{\delta=1}^{\tilde{k}} d_\delta \omega^{\delta}(t) = 0.$$

Taking this and (4,11) into account, we obtain

$$\sum_{\delta=1}^{\tilde{k}} d_\delta \mathbf{W}^{\delta}(t) = \lambda_1 \sum_{\delta=1}^{\tilde{k}} \vec{\Phi}_1^{\delta}(t) \left\{ \sum_{\beta=1}^{\tilde{k}} \left[C_{\beta\delta}^{(1)} + \lambda C_{\beta\delta}^{(2)} \right] d_\beta \right\} = 0 \qquad \left(\lambda = \frac{\lambda_2}{\lambda_1} \right).$$

Hence, by virtue of the linear independence of $\vec{\Phi}_1^{\delta}(t)$ $(\delta = 1, 2, \dots, \tilde{k})$, it follows that

$$\sum_{\beta=1}^{\tilde{k}} \left[C_{\beta\delta}^{(1)} + \lambda C_{\beta\delta}^{(2)} \right] d_\beta = 0 \quad (\delta = 1, 2, \dots, \tilde{k}). \qquad (4{,}14)$$

But this contradicts the assumption that not all d_δ are equal to zero.

From the linear independence of the system of functions (4.13) it follows that $\tilde{k} \leqslant k$, and, taking (4.7) into account, we obtain $k = \tilde{k}$. The theorem is proved.

Theorem 1 follows immediately from this theorem.

From Theorems 5 and 6 it follows:

The problem of solving the equation $T\varphi = f$ and the problem of solving the system of singular integral equations $D(t_0, t)\vec{\Phi}(t) = F(t_0)$ are equivalent.

Consider the system of equations adjoint to the homogeneous system (4.2)

$$D'(t, t_0)\, \tilde{x}(t) = 0, \qquad (4.15)$$

where $\tilde{x}(t)$ is the unknown vector. If we denote

$$\tilde{x}(t) = \{x_0(t), \alpha'(t) x_1(t), \dots, \alpha'_{n-1}(t) x_{n-1}(t)\}$$

and introduce the vector

$$x(t) = \{x_0(t), x_1(t), \dots, x_{n-1}(t)\},$$

then system (4.15) is rewritten in the form

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$$D^*(t, t_0)x(t) = 0. \tag{4.16}$$

It is easy to verify directly that system (4.16) is related to the equation $T'\psi = 0$ in exactly the same way as $D(t_0, t)\vec{\Phi}(t) = 0$ is related to $T\varphi = 0$. In particular, Theorem 6 is valid for them. Hence Theorems 2 and 3 follow immediately.

Taking into account the results obtained above, the proof of Theorem 4 presents no difficulty.

Let us note that if the operator T is not of normal type, Theorems 1-4, generally speaking, are not true.

From the arguments given above it is clear that if the normality of the operator T' follows from the normality of the operator T, then $\operatorname{ind} T = -\operatorname{ind} T'$.

If $\chi(\eta) = 0$, then equation (1.2) will be called quasi-Fredholm. For such equations all the Fredholm theorems are valid.

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REFERENCES CITED

1. N. I. Muskhelishvili, *Singular Integral Equations*, Moscow, 1962.
2. N. P. Vekua, *Systems of Singular Integral Equations*, Moscow-Leningrad, 1950.
3. G. S. Litvinchuk, Izv. Akad. Nauk SSSR, Ser. Mat., **25**, No. 6 (1961).
4. G. S. Litvinchuk, Dokl. Akad. Nauk, **149**, No. 5 (1963).
5. R. Isakhanov, Soobshch. Akad. Nauk GruzSSR, **20**, No. 1 (1958).

Note: Figure translations are in progress. See original paper for figures.

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