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Abstract

Full Text

MATHEMATICS

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SOLUTION OF CERTAIN MATRIX INEQUALITIES ARISING IN THE NONLINEAR THEORY OF CONTROL

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In recent years many problems in the theory of stability of motion (stability of equilibrium in the large, the existence of a periodic or almost periodic regime, their local or global stability, etc.) have been translated, in the works of many authors, into the language of Lyapunov functions. Determining necessary and sufficient conditions for the existence of Lyapunov functions is, in the general case, hardly a simpler problem than the original one. In the case, however, where special systems of differential equations are considered (for example, equations of automatic control) and a restricted class of Lyapunov functions (for example, quadratic forms), the problem of the existence of a Lyapunov function from this class usually becomes a purely algebraic problem on the existence of a solution of some special matrix inequality. Two such problems are considered in ⁽¹⁾. Below we give the solution of matrix inequalities to which the problem posed by M. A. Aizerman and F. R. Gantmacher in ⁽²⁾, p. 119, leads, as well as a number of analogous problems.

1°. Consider the system*

$$dx/dt = Px + q\varphi(\sigma), \quad \sigma = r^*x, \quad (1)$$

where P is a Hurwitz matrix; $\varphi(\sigma)$ is a continuous function satisfying the condition $0 \leq \sigma\varphi(\sigma) \leq \mu_0\sigma^2$ ($\mu_0 \leq +\infty$). Denote the set of all such functions by \mathfrak{M}_{μ_0} . For stability in the large of the solution $x \equiv 0$ it is sufficient (particular conditions of V. M. Popov ^(2,7)) that, for some ϑ and all $\omega \geq 0$, one have**

$$\pi(\omega) \equiv \mu_0^{-1} + \operatorname{Re}[(1+i\omega\vartheta)r^*(P-i\omega I)^{-1}] > 0, \quad +\infty > \lim_{\omega \rightarrow \infty} \omega^2\pi(\omega) > 0. \quad (2)$$

From the solution of the matrix inequalities in ⁽¹⁾ there follows a new proof of these conditions, and also the fact of the existence of a Lyapunov function of the form

$$\Omega(x) = x^* H x + \vartheta \int_0^\sigma \varphi(\sigma) d\sigma \quad (3)$$

when conditions (2) are fulfilled (see (5)). In connection with this there arises the problem of obtaining conditions for stability in the large for any $\varphi(\sigma) \in \mathfrak{M}_{\mu_0}$, encompassing

* Here and below, capital Latin letters denote $n \times n$ -matrices, lowercase Latin letters denote $n \times 1$ -vector columns (with the exception of t), and Greek letters denote numbers. Indices are denoted by arbitrary letters. Matrices, vectors, and numbers are real. The notation $H > 0$ ($H \geq 0$) means that $x^* H x > 0$ for $x \neq 0$ ($x^* H x \geq 0$), and I is the identity matrix. If $\mu_0 = +\infty$, then $\mu_0^{-1} = 0$.

** The formulation given here slightly refines Popov's original result (3). For further refinements (extension to critical cases, replacement of the sign $>$ in (2) by \geq , the possibility of the values $\pm\infty$ for the parameter ϑ), see (2,4). In (2) it is also shown that, when a simply verifiable necessary condition is satisfied, the second condition (2) may be discarded.

all conditions that can be obtained by means of the Lyapunov function (3) and thereby possibly improve conditions (2). In fact, it turns out that Popov's conditions (2) are such conditions. More precisely:

Theorem 1. Let $\mathfrak{M} \subset \mathfrak{M}_{\mu_0}$ be some class of continuous functions for which

$$\inf_{\sigma} \varphi(\sigma)/\sigma = 0, \quad \sup_{\sigma, \mathfrak{M}} \varphi(\sigma)/\sigma = \mu_0;$$

moreover, the \inf and \sup (for $\mu_0 \neq \infty$) are attained. a) For the existence of a matrix $H = H^*$ such that $\dot{\Omega} < 0$ for $x \neq 0$ and any $\varphi \in \mathfrak{M}$, it is necessary (for $\mu_0 < \infty$) and sufficient (for $\mu_0 \leq \infty$) that (2) be satisfied for all $\omega \geq 0$. b) If $\dot{\Omega} \leq 0$ for all $\varphi \in \mathfrak{M}$, then the first relation (2) is satisfied with the sign $>$ replaced by \geq , and, possibly, for the value $\vartheta = \pm\infty^*$.

Theorem 1 gives an answer to the problem posed by M. A. Aizerman and F. R. Gantmakher ((2), p. 119), since it shows that there do not exist absolutely stable systems (1) for which the fact of absolute stability can be established by means of a Lyapunov function of the form (3), but cannot be established by means of the S -procedure**.

From the results of ((2,5) (supplement) and Theorem 1 one can derive the following

Corollary. In the general case, when the spectrum of the matrix P lies in the left closed half-plane, the conditions of absolute stability in the class \mathfrak{M}_{μ_0} (or in the class $\mathfrak{M} \subset \mathfrak{M}_{\mu_0}$ indicated in Theorem 1) obtained under a suitable application of Lurie's method encompass all conditions that can be derived by means of Lyapunov functions of the form (3).

It also follows from Theorem 1 that, in the class of Lyapunov functions (3), the frequency condition

$$\mu_0^{-1} + \operatorname{Re} [(1 + i\omega\vartheta)r^*(P - i\omega I)^{-1}q] \geq 0$$

cannot be strengthened (i.e., replaced by a less restrictive one), for example, by introducing additional conditions of the form $|\varphi(\sigma)| \leq \varphi_0$, $|\varphi'(\sigma)| \leq \varphi_1$, etc. It can be shown, however, that Popov's frequency condition can be strengthened by simultaneously using conditions on the derivative $\varphi'(\sigma)$ and by extending the class of Lyapunov functions (3).

Let us also note the following generalization of Popov's condition, which is derived comparatively simply from (1):

Theorem 2. Let the spectrum of the matrix P be located in the domain

$$\operatorname{Re} \lambda < -\alpha_0 < 0, \quad \varphi(\sigma) \in \mathfrak{M}_{\mu_0}, \quad \mu_0 \neq \infty.$$

If, for some ϑ and all $\omega \geq 0$,

$$\mu_0^{-1} + \operatorname{Re}\{(1 + i\omega\vartheta)r^*[P - (i\omega - \alpha_0)I]^{-1}q\} > 0,$$

then for any $t \geq t_0$ and some $\alpha > \alpha_0$ one has

$$x(t) \leq \varkappa_0 |x(t_0)| \exp[-\alpha(t - t_0)],$$

where $x(t)$ is an arbitrary solution and the constant \varkappa_0 depends only on P, q, r, μ_0, α_0 .

2°. For given p, q, r, γ, P , where P is a Hurwitz matrix, and a matrix $H = H^*$, define

$$G = -(P^*H + HP), \quad g = -(Hq + p), \quad F(\mu, H) = G + \mu(gr^* + rg^*) + \gamma\mu^2 rr^*.$$

If H is the matrix in (3), then the condition $\dot{\Omega} < 0$ for all $\varphi \in \mathfrak{M}_{\mu_0}$ is equivalent, as is easy to verify, to the inequality $F(\mu, H) > 0$ for $0 \leq \mu \leq \mu_0$, where

$$p = \frac{\vartheta}{2} P^* r, \quad \gamma = -\vartheta r^* q.$$

(The last two relations are not assumed to hold.) The inequality $F(\mu, H) > 0$ also leads to the problem of finding necessary and sufficient conditions for the existence of Lyapunov functions in classes broader than (3).

Theorem 3. Let $\mu_0 \neq \infty$. For the existence of a matrix $H = H^*$ such that $F(\mu, H) > 0$ for $0 \leq \mu \leq \mu_0$, it is necessary and sufficient that, for some $\tau \geq 0$ and all $\omega \geq 0$, the following hold:

$$\pi_1(\omega) \equiv \gamma + 2 \operatorname{Re} p^*(P - i\omega I)^{-1}q + \tau [\mu_0^{-1} + \operatorname{Re} r^*(P - i\omega I)^{-1}q] > 0. \quad (4)$$

This condition is sufficient for $\mu_0 = \infty$, $\gamma \neq 0$. For $\mu = \infty$, $\gamma = 0$ it is neces-

* The latter means that the function $\operatorname{Re}[i\omega r^*(P - i\omega I)^{-1}q]$ does not take values of different signs.

** Condition (2), as shown by V. M. Popov (see (2)), encompasses all conditions of absolute stability in the class \mathfrak{M}_{μ_0} that can be obtained on the basis of the Lyapunov function (3) using the S-procedure. (For the S-procedure, see (2).)

necessary and sufficient conditions are the inequalities

$$\pi_1(\omega) > 0, \quad \lim_{\omega \rightarrow +\infty} \omega^2 \pi_1(\omega) > 0.$$

Theorem 4. For there to exist a matrix $H = H^*$ such that $F(\mu, H) \geq 0$ for $0 \leq \mu \leq \mu_0$, it is necessary that, for some $\tau \geq 0$ and all $\omega \geq 0$, $\pi_1(\omega) \geq 0$, where $\pi_1(\omega)$ is defined from (4). If the vectors $q, Pq, \dots, P^{\nu-1}q$ are linearly independent, then the latter condition is also sufficient.

Lemma. Let $a, b, c_h, \alpha_h, \rho_j$ ($j = 1, \dots, \chi$, $h = 1, \dots, \chi_1$) and a Hurwitz matrix P be given. Define the operator $Y = \mathfrak{A}(X)$ by the relation

$$P^*Y + YP = -X.$$

For the existence of χ real vectors u_j satisfying the equation

$$\mathfrak{A} \left(\sum_j u_j u_j^* + \sum_h \alpha_h c_h c_h^* \right) a + \sum_j \rho_j u_j + b = 0,$$

it is necessary that, for all $\omega \geq 0$,

$$\pi_2(\omega) \equiv \sum_j \rho_j^2 + 2 \operatorname{Re} b^* P_\omega^{-1} a - \sum_h \alpha_h |c_h^* P_\omega^{-1} a|^2 \geq 0, \quad (5)$$

where $P_\omega = P - i\omega I$. If the vectors $a, Pa, \dots, P^{\nu-1}a$ are linearly independent, then this condition is also sufficient.

Proof of the lemma. Let all $\alpha_h = 0$.

Necessity. By the hypothesis, we have

$$P_\omega^* U + U P_\omega = - \sum_j u_j u_j^*, \quad Ua + \sum_j \rho_j u_j + b = 0.$$

Multiplying the first relation on the left by $a^* P_\omega^{*-1}$ and on the right by $P_\omega^{-1} a$, the second on the left by $a^* P_\omega^{*-1}$, and comparing the values $\operatorname{Re} a^* P_\omega^{*-1} Ua$, we find $\pi_2(\omega) = \sum |\zeta_j - \rho_j|^2$, where $\zeta_j = a^* P_\omega^{*-1} u_j$, i.e. $\pi_2(\omega) \geq 0$.

Sufficiency. We shall seek the vectors u_j in the form $u_j = \rho_j \rho^{-1} u$, if $\rho^2 = \sum \rho_j^2 \neq 0$, and $u_j = u$, if $\sum \rho_j^2 = 0$. By Kalman's lemma ⁽⁶⁾, the vector u exists.

Let $\sum \alpha_h^2 \neq 0$. By what has been proved, the necessary and sufficient condition is the inequality $\pi_2(\omega) \geq 0$, where $\pi_2(\omega) = \sum \rho_j^2 + 2 \operatorname{Re} b_0^* P_\omega^{-1} a$, $b_0 = b + \mathfrak{A}(\sum \alpha_h c_{hc} h^*) a$. Since $\pi_2(\omega)$ is transformed into expression (5), the lemma is proved.

Proof of Theorem 3. Necessity, $\mu_0 \neq \infty$. From $F(0, H) = G > 0$ it follows that there exists a basis $r, u_1, \dots, u_{\nu-1}$ such that

$$G = \sum u_j u_j^* + \alpha r r^*, \quad \alpha > 0.$$

Representing g in the form $g = \sum \delta_j u_j + \beta/2 r$, we find

$$\mathfrak{A} \left(\sum_{j=1}^{\nu-1} u_j u_j^* + \alpha r r^* \right) q + \sum_{j=1}^{\nu-1} \delta_j u_j + \frac{\beta}{2} r + p = 0. \quad (6)$$

For any $x \neq 0$ we have

$$x^* F(\mu, H) x = \sum \xi_j^2 + 2\mu\eta \sum \delta_j \xi_j + (\alpha + \beta\mu + \gamma\mu^2)\eta^2 > 0,$$

where $\xi_j = x^* u_j$, $\eta = x^* r$. Hence it follows that

$$\delta = \sum |\delta_j|^2 < \alpha\mu^{-2} + \beta\mu^{-1} + \gamma \quad \text{for } 0 < \mu \leq \mu_0.$$

Two cases are possible: a) $\beta + 2\alpha\mu_0^{-1} \geq 0$, $\delta < \alpha\mu_0^{-2} + \beta\mu_0^{-1} + \gamma$, and b) $\beta + 2\alpha\mu_0^{-1} < 0$, $\delta < \gamma - \beta^2(4\alpha)^{-1}$.

In case a), from (6) we derive

$$\mathfrak{A} \left(\sum_{j=1}^{\nu} u_j u_j^* \right) q + \sum_{j=1}^{\nu-1} \delta u_j - \sqrt{\alpha} \mu_0^{-1} u_\nu + p + \tau \frac{r}{2} = 0,$$

where $u_\nu = \sqrt{\alpha} r$, $\tau = \beta + 2\alpha\mu_0^{-1}$, whence, by the lemma,

$$\alpha\mu_0^{-2} + \delta + \operatorname{Re}(2p + \tau r)^* P_\omega^{-1} q \geq 0.$$

From the inequality $\alpha\mu_0^{-2} + \delta < \gamma + \tau\mu_0^{-1}$, (4) follows.

In case b), from (6), by the lemma, we have

$$\delta + \operatorname{Re}(\beta r + 2p)^* P_\omega^{-1} q - \alpha |r^* P_\omega^{-1} q|^2 \geq 0.$$

Using the inequality $\delta < \gamma - \beta^2(4\alpha)^{-1}$, we find

$$\gamma + 2 \operatorname{Re} p^* P_\omega^{-1} q > (4\alpha)^{-1} |\beta - 2\alpha r^* P_\omega^{-1} q|^2,$$

i.e. (4) is satisfied for $\tau = 0$.

Let $\mu_0 = \infty$, $\gamma = 0$. From $F(\mu, H) > 0$ it follows easily that there exists $\tau \geq 0$ for which $g = \tau r/2$. By Theorem 2 ⁽¹⁾, the inequality $G > 0$ gives $\pi_1(\omega) > 0$, $\lim_{\omega \rightarrow \infty} \omega^2 \pi_1(\omega) > 0$.

Sufficiency. For $\mu_0 = \infty$, $\gamma = 0$, by Theorem 2 ⁽¹⁾ there exists $H = H^*$ satisfying the relations $G > 0$, $g = \tau r/2$, and, consequently, the inequality $F(\mu, H) > 0$. Let $\mu_0^{-1} + |\gamma| \neq 0$. Varying, if necessary, τ , we obtain $\pi_1(\omega) > 0$, $\gamma_0 \equiv \pi_1(\infty) = \gamma + \tau \mu_0^{-1} > 0$. Represent $F(\mu, H)$ in the form

$$F(\mu, H) = \gamma_0^{-1}(\gamma_0 G - g_0 g_0^*) + \gamma_0^{-1} f(\mu) f(\mu)^* + \tau \mu (1 - \mu \mu_0^{-1}) r r^*,$$

where $g_0 = g - \tau r/2$, $f(\mu) = g_0 + \gamma_0 \mu r$, and require that $\gamma_0 G - g_0 g_0^* > 0$. Then $F(\mu, H) > 0$ for $0 \leq \mu \leq \mu_0$. By Theorem 1 ⁽¹⁾, the required matrix $H = H^*$ exists. Theorem 3 is proved.

Theorem 4 is proved by the same scheme, with some complications.

Theorem 1 is a simple consequence of Theorems 3 and 4.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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