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Abstract

Full Text

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A VARIATIONAL METHOD FOR UNBOUNDED DOMAINS

(Presented by Academician I. M. Vinogradov on 16 XI 1963)

We shall consider the variational method for solving the first boundary-value problem for a self-adjoint elliptic equation in the case of an unbounded domain. This method had earlier been studied under various restrictions on the summability of the desired solution in the domain or on the summability of the boundary conditions (see, for example, ⁽¹⁻⁵⁾). Thus, already for the Laplace equation $\Delta u = 0$, the case of a constant solution was not included in the class of problems under consideration. The indicated restrictions are connected with the circumstance that the principal properties of the functional used in applying the variational method are the boundedness from below and lower semicontinuity of the functional. For unbounded domains, however, in contrast to bounded ones, the functionals under consideration are, generally speaking, **not bounded from below**, at any rate in the usual sense of the word.

In the present work a variational method is constructed without any restrictions on the summability of the boundary function, and only under the natural and, in a certain sense, minimal requirement of finiteness of the functional under consideration, which already ensures the existence and uniqueness of the solution of the problem. In part these results are based on embedding theorems for unbounded domains obtained earlier by the author ⁽⁶⁾.

The finiteness of the functional determines the natural class in which the solution of the variational problem is sought. For a quadratic functional, in the case of a bounded domain and bounded coefficients of the functional, a necessary and sufficient condition for its finiteness is that the function belong to some Sobolev class $W_2^{(r)}$, which also makes it natural to study the variational problems mentioned in these classes. The situation is quite different in the case of unbounded domains. The condition of finiteness of a quadratic functional is, generally speaking, not equivalent to the function's belonging to any space of the type $W_2^{(r)}$. We consider only the case of a half-space, since in it, on the one hand, all the specific features of the variational method for unbounded domains appear, while on the other hand the difficulties connected with the structure of the boundary of the domain disappear. It should be noted, however, that by the same method analogous results can also be obtained for unbounded domains of other types; for example, everything said above is transferred without difficulty to a domain that is the exterior of a ball.

Let E^n be the n -dimensional Euclidean space of points $x = (x_1, \dots, x_n)$;

$$E_+^n = \{x : x_n > 0\}; \quad E^{n-1} = \{x : x_n = 0\}; \quad \rho = \sqrt{x_1^2 + \dots + x_n^2};$$

$|y - x|$ is the distance between the points $x \in E^n$ and $y \in E^n$. We shall consider the bilinear functional

$$A(u, v) = \int \left[a^{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + b^i \left(\frac{\partial u}{\partial x_i} v + \frac{\partial v}{\partial x_i} u \right) + cuv \right] dE_+^n,$$

(where summation over the indices i and j is carried out from 1 to n), where $a^{ij} = a^{ji}$, $i, j = 1, 2, \dots, n$, are functions defined on E_+^n , and the corresponding quadratic functional is

$$A(u) = A(u, u).$$

Put

$$D_\alpha(u) = \int \frac{1}{(1 + \rho)^\alpha} \sum_{i=1}^n \left(\frac{\partial u}{\partial x_i} \right)^2 dE_+^n, \quad \alpha \geq 0.$$

Obviously, $D_0(u) = D(u)$ is the usual Dirichlet integral.

We shall assume that the functional $A(u)$ is a positive definite functional with weight $\frac{1}{(1 + \rho)^\alpha}$, namely, that there exists a constant $\beta > 0$ such that

$$A(u) \geq \beta D_\alpha(u). \quad (1)$$

If the coefficients of the functional $A(u)$ are such that there exists a constant $M > 0$ such that

$$|(1 + \rho)^\alpha a^{ij}| \leq M, \quad |(1 + \rho)^{\alpha+1} b^i| \leq M, \quad |(1 + \rho)^{\alpha+2} c| \leq M,$$

$$i, j = 1, 2, \dots, n,$$

and if $\alpha > n - 2$, then from the finiteness of the integral $D_\alpha(u)$ follows the finiteness of the integral $A(u)$ (although, of course, it is not in general estimated by the integral $D_\alpha(u)$); thus, in this case the functional $A(u)$ is finite if and only if the functional $D_\alpha(u)$ is finite.

Consider also the functional

$$K(u) = A(u) - 2(f, u). \quad (2)$$

If $A(u) < \infty$ and there exists $\alpha^* > n - 2$ such that

$$D_{\alpha^*}(u) < \infty,$$

$$\int (1 + \rho)^{\alpha^*+2} f^2 dE_+^n < \infty, \quad (3)$$

then the functional $K(u)$ is finite. Here one should keep in mind that if for a function u , for some $\alpha \geq 0$, $D_\alpha(u) < \infty$, then there certainly exists $\alpha^* > n - 2$ for which $D_{\alpha^*}(u) < \infty$; it suffices to take any α^* such that $\alpha^* \geq \alpha$ and $\alpha^* > n - 2$. In what follows we shall always assume that condition (3) is satisfied for some indicated α^* .

Finally, let

$$L(u) + f = 0 \quad (4)$$

be the Euler equation of the functional (2). Here

$$L(u) = \frac{\partial}{\partial x_i} \left(a^{ij} \frac{\partial u}{\partial x_j} \right) + qu, \quad q = -c + \frac{\partial b^i}{\partial x_i}, \quad i, j = 1, 2, \dots, n.$$

Let φ be some function defined on the hyperplane E^{n-1} . Denote by \mathfrak{D}_φ the set of all functions u for which

$$D_\alpha(u) < \infty, \quad u|_{E^{n-1}} = \varphi,$$

and by \mathfrak{R}_φ the set of all functions such that

$$A(u) < \infty, \quad u|_{E^{n-1}} = \varphi.$$

Obviously, by virtue of condition (1) we have \mathfrak{R}_φ . If the class \mathfrak{D}_φ is nonempty, then the function φ , being the boundary value of a function with finite integral $D_\alpha(u)$, satisfies the conditions (see (6))

$$\int_0^1 \frac{dh}{h^2} \int \frac{|\Delta_i(\varphi, h)|^2}{(1 + \rho)^\alpha} dE^{n-1} < \infty, \quad i = 1, 2, \dots, n - 1, \quad (5)$$

where $\Delta_i(\varphi, h)$ is the first difference of the function φ with step h in the i -th coordinate.

Conditions (5) are equivalent to the single condition

$$\int \frac{dE_x^{n-1}}{(1+\rho)^\alpha} \int_{|y-x| \leq 1} \frac{|\varphi(y) - \varphi(x)|^2}{|y-x|^n} dE^{n-1} < \infty. \quad (6)$$

If, however, the function φ satisfies the conditions

$$\int_0^{+\infty} \frac{dh}{h^2} \int \frac{|\Delta_i(\varphi, h)|^2}{(1+\rho)^\alpha} dE^{n-1} < \infty, \quad i = 1, 2, \dots, n-1, \quad (7)$$

then it can be extended to E^n as a function u with finite integral $D_\alpha(u)$. In the case $\alpha = 0$, conditions (5) and (7) are equivalent.

Thus, the following holds.

Theorem 1. *In order that the class \mathfrak{K}_φ be nonempty, it is necessary that conditions (7) be satisfied. Moreover, from the fulfillment of conditions (7) it follows (see (6)) that the function φ is summable on E^{n-1} with weight*

$$\frac{1}{(1+\rho)^{\alpha+1}}.$$

Theorem 2. *If the class \mathfrak{K}_φ is nonempty, then the functional $K(u)$ is bounded below on \mathfrak{K}_φ , and there exists, moreover a unique, function $u_0 \in \mathfrak{K}_\varphi$ giving the minimum of the functional $K(u)$ in the class \mathfrak{K}_φ .*

In the proof of Theorem 2 an essential role is played by the following lemma, which replaces the usual semiboundedness of the functional:

Lemma. *Let $\alpha > n-2$ and $u|_{E^{n-1}} = 0$; then there exists a constant $\gamma > 0$ such that*

$$\int \frac{u^2 dE^n}{(1+\rho)^{\alpha+2}} \leq \gamma D_\alpha(u).$$

Theorem 3. *If the class \mathfrak{K}_φ is nonempty, then the function u_0 giving the minimum of the functional $K(u)$ in the class \mathfrak{K}_φ is a generalized solution of Euler's equation (4), i.e.*

$$A(u, \delta u) - (f, \delta u) = 0,$$

where δu is any smooth finite function. In the case of sufficient smoothness of the coefficients of the functional $A(u)$, the generalized solution in the class \mathfrak{K}_φ of equation (4) is unique.

Sufficient smoothness of the coefficients is understood here in the sense that it ensures the existence of a solution $u \in W_2^{(2)}$ (for notation see (1)) of the equation $L(u) = \delta$ with zero boundary conditions in any finite domain.

Theorem 4. *Let the coefficients of Euler's equation (4) be such that the derivatives $\partial a^{ij}/\partial x_k$, $i, j, k = 1, 2, \dots, n$, are square-summable in any finite domain and*

$$|a^{ij}| \leq \frac{A}{(1+\rho)^\alpha}, \quad |b^i| \leq \frac{A}{(1+\rho)^{\alpha^*+1}}, \quad i, j = 1, 2, \dots, n, \quad \alpha^* \geq \alpha \geq 0, \quad \alpha^* > n-2, \quad (8)$$

A being a constant. Suppose, further, that the generalized solution u_0 of Euler's equation (4) has second generalized derivatives square-summable in any finite domain. Then u_0 is an ordinary solution of the problem (i.e. satisfies equation (4) almost everywhere), and this solution is unique in the class of functions belonging to \mathfrak{K}_φ and having second generalized derivatives square-summable in any finite domain.

For the proof of this theorem, Green's formula is of essential importance,

$$A(u, v) = -(L(u), v),$$

obtained under the assumptions that conditions (8) are fulfilled, that $D_\alpha(u) < \infty$, $D_\alpha(v) < \infty$, $v|_{E^{n-1}} = 0$, and that the integral $(L(u), v)$ exists.

Let us formulate the corollaries that are obtained from the results presented for the case when the coefficients of the Euler equation (4) are constant and the solution is sought in the class of functions having a finite ordinary Dirichlet integral

$$D(u) < \infty \quad (9)$$

and assuming on the hyperplane E^{n-1} the prescribed value:

$$u|_{E^{n-1}} = \varphi.$$

We denote this class of functions by \mathfrak{D}_φ^0 . Consider, as an example, the Laplace equation

$$\Delta u = 0.$$

Noting that from (9) it follows that

$$D_{n-2+\varepsilon}(u) < \infty$$

for any $\varepsilon > 0$, and applying the theorems given above, we obtain:

Theorem 5. *In order that the class \mathfrak{D}_φ^0 be nonempty, it is necessary and sufficient that*

$$\int dE_x^{n-1} \int_{|y-x|\leq 1} \frac{|\varphi(y) - \varphi(x)|^2}{|y-x|^n} dE_y^{n-1} < \infty$$

and that the function φ be square-summable with weight

$$\frac{1}{(1+\rho)^{n-1+\varepsilon}}, \quad \varepsilon > 0,$$

the second condition following from the first. In every nonempty class \mathfrak{D}_φ^0 there exists, moreover uniquely, a harmonic function.

Let us note that, as is clear from the simplest examples, one cannot put $\varepsilon = 0$ here.

This theorem gives a definitive answer to the question of the existence and uniqueness of a harmonic function in the class \mathfrak{D}_φ^0 . Previously this question had been solved only under additional restrictions. For example, in the author's paper ⁽³⁾ (p. 167) there is a statement that in every nonempty class \mathfrak{D}_φ^0 there exists, moreover uniquely, a harmonic function. Criteria for nonemptiness of the class \mathfrak{D}_φ^0 , however, had earlier been obtained only under the assumption of one or another summability of the boundary function ⁽²⁻⁵⁾.

The main results of the present article are a certain strengthening of results reported by the author at the Soviet-American Symposium on Partial Differential Equations in Novosibirsk in August 1963 ⁽¹³⁾.

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Note: Figure translations are in progress. See original paper for figures.

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