FINITE GROUPS WITH DISPERSIVE SECOND MAXIMAL SUBGROUPS

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Abstract

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Ya. G. BERKOVICH

FINITE GROUPS WITH DISPERSIVE SECOND MAXIMAL SUBGROUPS

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§ 1. The aim of this paper is a complete description of those nonsolvable groups in which all second maximal subgroups have invariant 2-complements. From this result, in particular, one obtains a description of nonsolvable groups with dispersive second maximal subgroups (here the ordering of the set of primes with respect to which dispersivity is considered is taken in increasing order). Such a description has become possible thanks to the work of Thompson and Feit, who proved the solvability of groups of odd order (^{1,2}), of Gorenstein and Walter (^{3}) on the structure of groups with a dihedral Sylow 2-subgroup, and also Thompson' s theorem (unpublished), which gives a complete description of minimal simple groups (a simple group is called minimal if all its proper subgroups are solvable)*. More briefly: the aim of the paper is to solve Question 1 posed in § 9 of (^{15}).

For the reader' s convenience we state the above-mentioned results of Gorenstein –Walter (^{3}) and Thompson.

Theorem A (D. Gorenstein and J. Walter). Let G be a finite simple group of order $4g'$ with odd g', and let its Sylow 2-subgroup coincide with its centralizer. Then $G \cong LF(2, q)$, where q is a power of a prime congruent to 3 or 5 modulo 8.

Theorem B (J. Thompson). Let G be a minimal nonabelian simple group. Then G is isomorphic to one of the following groups:

(1) $LF(2, 2^p)$, where p is any prime number.

(2) $LF(2, 3^p)$, where p is any odd prime number.

(3) $Sz(2^p)$, where p is any odd prime number; $Sz(q)$ is the Suzuki group (^{4,5}).

(4) $LF(2, p)$, where p is any prime number greater than 3 and such that

$$p^2 + 1 = 0 \pmod 5.$$

(5) $LF(3, 3)$.

Only finite groups are considered.

Let us recall some definitions (6). We shall say that a group G has an **invariant p-complement** if some homomorphic image of it is isomorphic to a Sylow p-subgroup of the group (here p is a prime number). In particular, a group whose order is not divisible by p also has an invariant p-complement.

Let φ be some ordering of the set of all prime numbers (all orderings considered here will be total). The ordering φ induces a completely determined ordering on any set of prime numbers (we shall agree to denote this induced ordering by the same letter φ). We shall call a prime number p **φ-minimal divisor** of the order of the group G, if all divisors of the order of the group G distinct from p follow p in the ordering φ. We shall call a group G φ-dispersive if every subgroup H of it (H may coincide with G) has an invariant p-complement for the φ-minimal prime divisor p of the order of H (cf. with (6)). The ordering in increasing order will be denoted by the letter v.

§ 2. **Theorem 1.** *Let all second maximal subgroups of a nonsolvable group G have invariant 2-complements. Then the group G is isomorphic to one of the following groups:*

1. $LF(2, 2^p)$, where $2^p - 1$ is a Mersenne prime.
2. $LF(2, 3^p)$, where p is any odd prime.
3. $LF(2, p)$, where p is any prime > 3 such that $p^2 \equiv 9 \pmod{80}$.
4. $SL(2, p)$, where p is the same prime as in (3).
5. $SL(2, 3^p)$, where p is any odd prime.
6. $SL(2, 5)$.

Proof. Denote by the letter Φ the Frattini subgroup of the group G. From the remark on p. 584 of (15) it follows that G/Φ is a minimal simple group of composite order. Indeed, by Theorem 1 of (6), the maximal subgroups of the group G either have invariant 2-complements (and therefore, by (1,2), are solvable), or are groups of type S (7–9) (which are also solvable). If $\Phi = 1$, then the result follows immediately from Theorem B (the simple verification that is necessary here we omit). Therefore let $\Phi \neq 1$. Since the property possessed by the group G is preserved under homomorphism, G/Φ is one of the simple groups mentioned in the conclusion of Theorem 1.

Among the proper subgroups of the group G there is a subgroup H that has no invariant 2-complement (otherwise G, by (6,1,2), would be solvable). Therefore the subgroup H is maximal in the group G. It follows that all proper subgroups of the group H have invariant 2-complements (while H itself has no such complement); hence H, by Theorem 1 of (6), is of type S with an invariant Sylow

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[*] Thompson' s theorem was formulated in the survey lecture of A. I. Kostrikin at the Fifth All-Union Colloquium on General Algebra in Novosibirsk in 1963.

2-subgroup. Put $(H) = 2^{\alpha} q^{\beta}$, where (H) denotes the order of H: $\alpha, \beta \geq 1$; $q > 2$ is a prime (in what follows, the letter H denotes precisely this subgroup). Consider all possible cases that may arise.

(a) $G/\Phi \simeq LF(2, 2^p)$, $2^p - 1$ is a Mersenne prime.

If T is a group, then T_q will denote its Sylow q-subgroup. Since the group G/Φ_2 satisfies the condition of the theorem, in order to prove the equality $\Phi_q = 1$ it suffices for the moment to put $\Phi_2 = 1$. Since $G_q = H_q$ and H is of type S with a noninvariant Sylow q-subgroup, G_q is a cyclic subgroup. As is known from Wielandt' s theorem (^{10}), a group with a cyclic Sylow q-subgroup is either q-solvable or q-simple. Since the group G is not q-solvable (for even its factor group G/Φ is not q-solvable), it is q-simple, and therefore $\Phi_q = 1$. Thus Φ is a 2-subgroup. If p is odd, then the theorem of Yu. A. Gol' fand (^{9}) gives $a = p$, which is impossible if $\Phi \neq 1$. Therefore $p = 2$. In this case $q = 3$, and the theorem of Yu. A. Gol' fand (^{9}) gives $a = 3$. Since $G/\Phi \simeq LF(2,4) \simeq LF(2,5)$ and the order of Φ is 2, the application of Schur' s theorem $((^{11}), p.120)$ gives $G \simeq SL(2,5)$.

(b) $G/\Phi \simeq LF(2, p)$, $p^2 \equiv 9 \pmod{80}$.

In this case H/Φ is a tetrahedral group, and $q = 3$. Again, as in (a), in order to prove the equality $\Phi_3 = 1$, suppose for the moment that $\Phi_2 = 1$. From the already cited theorem of Wielandt on a group with a cyclic Sylow subgroup (^{10}) it follows that G_3 is a noncyclic group. Then in G_3 there is a subgroup P of order 3 that is not contained in Φ_3 (Φ_3 is cyclic, since it is contained in H_3). But then the subgroup $P\Phi_3$ is noncyclic. Since in G/Φ_3 all subgroups of order 3 are conjugate, $P\Phi_3/\Phi_3$ is conjugate to some subgroup of H/Φ_3 (denote this subgroup—

than through T/Φ_3. But then $P\Phi_3$ and T are conjugate in G, which is impossible, since $P\Phi_3$ is noncyclic, while T is cyclic. Thus, in G_3 there is no subgroup such as P. This is a contradiction, proving the equality $\Phi_3 = 1$. Consequently, Φ is a 2-subgroup. Since $q = 3$, Yu. A. Gol' fand' s theorem $^{(9)}$ gives $\alpha = 3$. In this case Φ has order 2. Application of Schur' s theorem $(^{(11)}$, p. 120) shows that G is isomorphic to $SL(2, p)$.

(c) $G/\Phi \simeq LF(2, 3^p)$, where p is any odd prime. Note that in this case the order of the group G/Φ is not divisible by 8.

In this case H/Φ is a tetrahedral group. Exactly as in (a) and (b), it is shown that the subgroup Φ has order 2 (here a completely analogous method is used, as well as the fact that in $LF(2, 3^p)$ all subgroups of order 3 are conjugate). Since the subgroup H_2 is nonabelian, it contains a cyclic subgroup of order 4 (see $^{(9)}$). Using the method of item (b) and the fact that in G/Φ all subgroups of order 2 are conjugate, we shall show that Φ is the unique subgroup of order 2 in G (and therefore also in G_2). Since the group G has no invariant 2-complement, its Sylow 2-subgroup G_2 cannot be cyclic. Therefore G_2 is an ordinary quaternion group. It is now clear that $G \simeq SL(2, 3^p)$. The theorem is proved.

Corollary 1. *Let G be a nonsolvable group with v-dispersive second maximal subgroups. Then it is isomorphic to one of the groups (1)–(4), (6) listed in the conclusion of Theorem 1.*

Groups of types (1)–(4), (6) do indeed satisfy the condition of the corollary. A group of type (5), however, does not satisfy the condition of the corollary, since it contains a subgroup of order $3^p(3^p-1)$, which is not v-dispersive and is not of type S. But this contradicts Theorem 1 from (6).

Now the following is completely obvious.

Corollary 2. *Let G be a nonsolvable group with supersolvable second maximal subgroups. Then it is isomorphic to one of the groups (1), (3), (4), (6) of Theorem 1, and also to the fractional-linear groups $LF(2,3^p)$ for which the number 3^p-1 is equal to twice a prime.*

§ 3. Using Theorem A, the Brauer-Suzuki theorem on the nonsimplicity of a group with a quaternion Sylow 2-subgroup (14), Theorem 1, and also the methods of (15), the following result is proved:

Theorem 2. *Let every solvable subgroup H of a nonsolvable group G be such that all proper subgroups of H have invariant 2-complements. Then either G contains a proper subgroup $LF(2,2^p)$ ($2^p-1\geq 7$ is a Mersenne prime; in this case the group G is simple, and $2^p(2^p-1)$ is the order of the normalizer of its Sylow 2-subgroup), or G is isomorphic to one of the groups (1)–(6) listed in Theorem 1. Moreover, in (3) and (4) the number p is congruent to 3 or 5 modulo 8.*

We do not know a single example of a group from Theorem 2 that would contain a proper subgroup $LF(2,2^p)$, where $2^p-1\geq 7$ is a Mersenne prime. It is clear that Theorem 2 is a generalization of Theorem 1. From Theorem 2 there also follow analogous results of (12,13), as well as Theorems 7, 15, 16 of (15).

Institute of Mathematics and Computing Technology
Academy of Sciences of the BSSR

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Note: Figure translations are in progress. See original paper for figures.

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