MATHEMATICAL PHYSICS

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Abstract

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ON OPERATORS IN THE REPRESENTATION OF SECOND QUANTIZATION

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1. Let M be some set endowed with a measure; $x \in M$; $\hat{a}(x)$, $\hat{a}^*(x)$ are operator distributions satisfying the canonical (Bose or Fermi) commutation relations. We shall regard the operator distributions $\hat{a}(x)$, $\hat{a}^*(x)$ as realized in the usual way in the space of states $\mathcal{H}$, in which there exists a vacuum vector $\hat{\Phi}_0$.

The operator distributions $\hat{a}(x)$, $\hat{a}^*(x)$ play the role of generators in the algebra of operators of the space $\mathcal{H}$. Recall that the normal form of an arbitrary operator $\hat{A}$ in $\mathcal{H}$ is the expression of it in terms of the generators $\hat{a}(x)$, $\hat{a}^*(x)$ in the form

$$\hat{A} = \sum \int K_{mn}(x_1 \dots x_m \mid y_1 \dots y_n)\, \hat{a}^*(x_1) \dots \hat{a}^*(x_m) \hat{a}(y_1) \dots \hat{a}(y_n)\, d^m x\, d^n y. \quad (1)$$

In the present paper we study some simplest properties of operators represented in the form (1).

2. Let us make precise the definition of the normal form of an operator. Let $\hat{\Phi} \in \mathcal{H}$:

$$\hat{\Phi} = \sum \frac{1}{\sqrt{m!}} \int K_m(x_1 \dots x_m)\, \hat{a}^*(x_1) \dots \hat{a}^*(x_m)\, d^m x\, \hat{\Phi}_0. \quad (2)$$

We shall call the vector $\hat{\Phi}$ **finite** if the sum (2) contains only a finite number of terms.

We shall call the operator $\hat{A}$ **representable in normal form** if it can be represented in the form of the series (1), strongly convergent on a dense set D consisting of finite vectors (D need not consist of all finite vectors).

Theorem 1. *Every bounded operator in $\mathcal{H}$ is representable in normal form.*

The space $\mathcal{H}$ decomposes into the direct sum of subspaces $\mathcal{H}_m$, consisting of vectors of the form (2), for which, however, the sum standing on the right-hand side of (2) consists of a single term. In accordance with this, every operator in $\mathcal{H}$ is naturally specified by means of a matrix $\|A_{mn}\|$, the elements of which are operators from $\mathcal{H}_n$ to $\mathcal{H}_m$. The space $\mathcal{H}_n$ is naturally interpreted as the space of functions $K(x_1, \dots, x_n)$, symmetric in the Bose case and antisymmetric in the Fermi case. Correspondingly, the operators A_{mn} will be specified by kernels $A_{mn}(x_1 \dots x_m \mid y_1 \dots y_n)$; the functions A_{mn} (generally speaking, generalized) will be assumed in the Bose case to be symmetric separately in $x_1 \dots x_m$ and separately in $y_1 \dots y_n$, and in the Fermi case antisymmetric in these variables.

Let us introduce the functions $a(x)$, $a^*(x)$ into consideration. In the Bose case $a(x)$, $a^*(x)$ are complex-conjugate functions with summable square; in the Fermi case they are functions with anticommuting values.

Let us associate with each operator $\hat{A}$, written in normal form, the functional

$$A(a^*, a) =$$

$$= \sum \int K_{mn}(x_1 \dots x_m \mid y_1 \dots y_n)\, a^*(x_1) \dots a^*(x_m) a(y_1) \dots a(y_n)\, d^m x\, d^n y, \quad (3)$$

where K_{mn} are the same functions as in (1).

With the matrix of the same operator $\|A_{mn}\|$ we associate the functional

$$\widetilde{A}(a^*, a) =$$

$$= \sum \frac{1}{\sqrt{m!n!}} \int A_{mn}(x_1 \dots x_m \mid y_1 \dots y_n)\, a^*(x_1) \dots a^*(x_m)\, a(y_1) \dots a(y_n)\, d^m x\, d^n y. \quad (4)$$

It is obvious that, knowing the functionals (3) or (4), one can reconstruct the operator $\hat{A}$.

Theorem 2. Let $\hat{A}$ be a bounded operator. Then, in the Bose case, the series (3) and (4) converge absolutely and uniformly for all functions $a(x)$ satisfying the condition $\int |a(x)|^2\, dx < R^2$, $R > 0$ being an arbitrary number.

In the Fermi case the series (3) and (4) will be understood as formal.

Theorem 3. The functionals (3) and (4) are connected by the relation

$$A(a^*, a) = \widetilde{A}(a^*, a) \exp\left[-\int a^* a\, dx\right]. \quad (5)$$

This formula is valid both in the Fermi and in the Bose case.

Let us associate with each vector $\hat{\Phi} \in \mathcal{H}$ the functional

$$\Phi(a^*) = \sum \frac{1}{\sqrt{m!}} \int K_m(x_1 \dots x_m)\, a^*(x_1) \dots a^*(x_m)\, d^m x. \tag{6}$$

By $\Phi^*(a)$ we denote the functional

$$\Phi^*(a) = \sum \frac{1}{\sqrt{m!}} \int \overline{K_m}(x_1 \dots x_m)\, a(x_m) \dots a(x_1)\, d^m x, \tag{6'}$$

where K_m are the same functions as in (2); $a(x)$, $a^*(x)$ in the Bose case are complex-conjugate functions, and in the Fermi case are functions with anticommuting values. We note that the scalar product in $\mathcal{H}$ can be given by the formula*

$$(\hat{\Phi}, \hat{\Phi}) = \int \Phi^*(a)\, \Phi(a^*) \exp\left[-\int a^* a\, dx\right] \prod da^* da. \tag{7}$$

On the right-hand side of (7) stands a continual integral: in the Bose case an ordinary one, and in the Fermi case one over anticommuting variables (for the definition of the integral over anticommuting variables see ([1, 3])).

Theorem 4. The action of an operator on a vector and the product of operators can be given by the formulas:

$$\hat{A}\hat{\Phi} = \hat{\Psi} \leftrightarrow \Psi(a^*) = \int \widetilde{A}(a^*, a)\, \Phi(a^*) \exp\left[-\int a^* a\, da\right] \prod da^* da,$$

$$\hat{A}\hat{B} = \hat{C} \leftrightarrow \widetilde{C}(a^*, a) = \int \widetilde{A}(a^*, a)\, \widetilde{B}(a^*, a) \exp\left[-\int a^* a\, da\right] \prod da^* da. \tag{8}$$

Both formulas are valid both in the Bose and in the Fermi cases. The continual integrals on the right-hand side of (8) are, in the Fermi case, integrals over anticommuting variables.

In formulas (8) there appear functionals corresponding to the matrix representation of the operators A, B, C. Using formula (5), it is not difficult to find the expression for the vector $\hat{\Psi} = \hat{A}\hat{\Phi}$ and the operator $\hat{C} = \hat{A}\hat{B}$ through the functionals corresponding to the normal form of the operators $\hat{A}$, $\hat{B}$, $\hat{C}$.

* See ([1, 2]). In ([2]) this theorem is proved for the Bose case.

3. Consider the expression

$$\hat{H} = \frac{1}{2}\int [A(x,y)\hat{a}^*(x)\hat{a}^*(y) + 2C(x,y)\hat{a}^*(x)\hat{a}(y) + \overline{A}(x,y)\hat{a}(x)\hat{a}(y)]\,dxdy +$$

$$+\int [f(x)a^*(x) + f^*(x)a(x)]\,dx, \tag{9}$$

where $A(x,y), C(x,y)$ are the kernels of the operators A, C in $L_2(M)$, with C a self-adjoint operator, and $f(x)$ is a certain generalized function. In the fermion case $f = 0$.

We shall say that $\hat{H}$ has **operator meaning** if there exists at least one vector $\hat{\Phi} \neq 0$ for which $\hat{H}\hat{\Phi} \in \mathcal{H}$.

Denote by C_1 the operator in the space of operators defined by the formula $C_1 A = CA + AC$.

Theorem 5. In order that expression (9) have operator meaning, it is sufficient that, for at least one complex z, the inequalities

$$\mathrm{sp}\,[((C_1 - z)^{-1}A)A^*] < \infty, \qquad ((C - z)^{-1}f, f) < \infty \tag{10}$$

hold.

It is very likely that conditions (10) are also necessary in order that expression (9) have operator meaning.*

Theorem 6. If inequalities (10) hold, the operator (9) is self-adjoint.

Let T be a certain operator in $L_2(M)$, given by the kernel $T(x,y)$. Denote by $\overline{T}$ and T' the operators given by the kernels $\overline{T}(x,y)$ and $T(y,x)$, respectively.

Consider the operators Φ, Ψ in $L_2(M)$ defined by the equality

$$\begin{pmatrix} \Phi & \Psi \\ \overline{\Psi} & \overline{\Phi} \end{pmatrix} = \exp\left\{ it \begin{pmatrix} -C & -A \\ \pm\overline{A} & \overline{C} \end{pmatrix} \right\},$$

where the upper sign is taken in the Bose case and the lower sign in the Fermi case.

Theorem 7. If inequality (10) holds, the matrix of the operator $\hat{U} = e^{it\hat{H}}$ is given by the functional

$$\tilde{U}(t, a^*, a) =$$

$$= c\exp\left\{ \pm\frac{1}{2}(a, a^*) \begin{pmatrix} \overline{\Psi}\Phi^{-1} & \Phi'^{-1} \\ \pm\Phi^{-1} & -\Phi^{-1}\Psi \end{pmatrix} \begin{pmatrix} a \\ a^* \end{pmatrix} + a(g^* - \overline{\Psi}\Phi^{-1}g) - a^*\Phi^{-1}g \right\},$$

$$c = (\det \Phi e^{iCt})^{\mp 1/2} \exp \left\{ i \int_0^t (g\Phi'^{-1}\overline{A}\Phi^{-1}g - f^*\Phi^{-1}g)\, dt \right\},$$

$$\begin{pmatrix} g \\ g^* \end{pmatrix} = -i \int_0^t \begin{pmatrix} \Phi & \Psi \\ \overline{\Psi} & \overline{\Phi} \end{pmatrix} \begin{pmatrix} -f \\ f^* \end{pmatrix} dt.$$

In the fermion case $f = g = 0$. The upper sign in these formulas refers to the Bose case, the lower to the Fermi case. For brevity, the sign of integration over the set M is omitted everywhere; for example,

$$ag^* = \int a(x) g^*(x,t)\, dx.$$

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CITED LITERATURE

1. F. A. Berezin, R. A. Minlos, L. D. Faddeev, Proc. IV All-Union Mathematical Congress, 1961.

2. V. Bargmann, Proc. Nat. Acad. Sci. USA, 48, No. 2 (1962).

3. F. A. Berezin, DAN, 137, No. 2 (1961).

* We note that if C is the operator of multiplication by a function $c(x)$, then conditions (10) can be written i

$$\int \frac{|A(x,y)|^2}{|c(x)+c(y)-z|}\, dx\, dy < \infty, \qquad \int \frac{|f(x)|^2}{|c(x)-z|}\, dx < \infty.$$

Note: Figure translations are in progress. See original paper for figures.