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Abstract

Full Text

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CLOSED ORTHONORMAL SYSTEMS OF ANALYTIC FUNCTIONS IN FINITELY CONNECTED DOMAINS

(Presented by Academician V. I. Smirnov on 20 IV 1964)

Notation. Let B be a finitely connected domain of the z-plane, containing the point at infinity, with boundary K, consisting of m closed mutually exterior analytic Jordan curves $K^{(1)}, K^{(2)}, \dots, K^{(m)}$; $\overline{B} = B \cup K$; $G(z, \infty, B)$ is the Green function for the domain B with pole at $z = \infty$; K_ρ $(1 \le \rho < \infty)$ is a level line of the Green function, i.e. $K_\rho = E(G(z, \infty; B) = \ln \rho)$; B_ρ is the domain containing $z = \infty$ and bounded by the level line K_ρ; $B_{r,R}$ $(1 \le r < R)$ is the annular set $E(\ln r < G(z, \infty; B) < \ln R)$.

We consider the following classes of functions: $\sigma(B)$ is the class of functions $f(z)$, regular* in the domain B, $f(\infty) = 0$, for which

$$\iint\limits_B |f'(z)|^2 \, d\sigma < \infty;$$

$l^2(B)$ is the class of functions $f'(z)$, where $f(z) \in \sigma(B)$; $\Sigma(B)$ is the class of meromorphic and single-valued functions $F(z)$ in the domain B, having a single pole at the point at infinity and a Laurent expansion in its neighborhood of the form $F(z) = z + \alpha_0 + \alpha_1 z^{-1} + \cdots$; $\widetilde{\Sigma}(B)$ is the subclass of $\Sigma(B)$ of functions $F(z)$ mapping B onto domains without exterior points and with zero boundary area.

I. Laurent system of functions

1. Existence and uniqueness of the Laurent system

We shall call a Taylor system of functions for the domain B a system of functions $\{\varphi_n(z)\}$ $(n = 1, 2, \dots)$ satisfying the conditions: a) the functions $\varphi_n(z)$ $(n = 1, 2, \dots)$ are regular in the closed domain $\overline{B}$ and have, in a neighborhood of $z = \infty$, a Taylor expansion of the form

$$\varphi_n(z) = \sum_{k=n}^{\infty} a_{nk} z^{-k}, \qquad a_{nn} > 0 \quad (n = 1, 2, \dots); \tag{1}$$

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b) the derivatives $\varphi_n'(z)$ $(n = 1, 2, ...)$ form an orthonormal system in the domain B, i.e.

$$\frac{1}{\pi} \iint\limits_B \varphi_n'(z)\, \overline{\varphi_k'(z)}\, d\sigma = \delta_{nk} \quad (n, k = 1, 2, ...),$$

complete in the class $l^2(B)$.

We shall call the conjugate system of functions (for the Taylor system of the domain B) a system of functions $\{\Phi_n(z)\}$ $(n = 1, 2, ...)$ satisfying the conditions: a) the functions $\Phi_n(z)$ $(n = 1, 2, ...)$ are regular in the closed domain $\overline{B}$ except for a pole of order n at the point $z = \infty$, and their Laurent expansion in a neighborhood of $z = \infty$ contains no constant term; b) on each boundary curve $K^{(\nu)}$ $(\nu = 1, 2, \dots, m)$ the relation

$$\Phi_n(z) = \overline{\varphi_n(z)} + K_{n\nu} \quad (n = 1, 2, \dots;\ \nu = 1, 2, \dots, m), \tag{2}$$

holds, where $K_{n\nu}$ is a constant.

Finally, by a Laurent system for the domain B we shall mean a system of pairs of functions

$$\{\varphi_n(z), \Phi_n(z)\} \quad (n = 1, 2, ...).$$

* By functions regular (meromorphic) in $\overline{B}$ we mean single-valued regular (meromorphic) functions.

We shall now set forth some of the results obtained that are used for the study of univalent functions.

Theorem 1. *For the given domain B, the Taylor and Laurent systems of functions exist and are unique.*

The proof uses a well-known result of Grunsky ([1]) stating that for every polynomial $P(z)$ there exists a unique function $u(z) = P(z) + f(z)$, where $f(z)$ is regular in $\overline{B}$, $f(\infty) = 0$, which maps the boundary curves $K^{(\nu)}$ $(\nu = 1, 2, \dots, m)$ one-to-one onto segments of parallel straight lines of a prescribed inclination to the real axis. The existence of the Taylor system was proved by Bergman ([2]). Systems of functions close to the Taylor system were considered by Walsh ([3]). If the domain B is the exterior of the unit circle, then $\varphi_n(z) = \dfrac{1}{\sqrt{n}} z^{-n}$, $\Phi_n(z) = \dfrac{1}{\sqrt{n}} z^n$ $(n = 1, 2, ...)$.

2. Properties of the functions of the Laurent system.

a) For the functions of the Laurent system the following equalities hold

$$\frac{i}{2\pi} \oint_K \varphi_k(z)\, d\overline{\varphi_n(z)} = \delta_{kn}, \qquad \frac{i}{2\pi} \int_K \varphi_k(z)\, d\varphi_n(z) = 0 \quad (n, k = 1, 2, ...), \tag{3}$$

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$$\frac{i}{2\pi}\oint_K \Phi_k(z)\,d\overline{\Phi_n(z)} = -\delta_{kn}, \qquad \frac{i}{2\pi}\int_K \Phi_k(z)\,d\Phi_n(z) = 0 \quad (n,k=1,2,...). \tag{4}$$

b) For the functions of the Taylor system $\{\varphi_n(z)\}$, for each z in the domain B we have:

$$\lim_{n\to\infty}\left|\varphi_n^{(\nu)}(z)\right|^{1/n} = e^{-G(z,\infty,B)} = \rho^{-1}, \qquad z\in B \quad (\nu=0,1,...). \tag{5}$$

Moreover, for any $\varepsilon>0$ the inequality holds

$$\left|\varphi_n^{(\nu)}(z)\right| \le M_\nu(1+\varepsilon)^{n+\nu}\rho^{-(n+\nu)}, \qquad z\in\overline{B} \quad (n=1,2,...;\ \nu=0,1,...), \tag{6}$$

where M_ν depends only on ε and ν.

c) For the functions of the conjugate system $\{\Phi_n(z)\}$, for each finite z in the domain B we have:

$$\lim_{n\to\infty}\left|\Phi_n^{(\nu)}(z)\right|^{1/n} = e^{G(z,\infty;B)} = \rho, \qquad z\in B_{1,\infty} \quad (\nu=1,2,...). \tag{7}$$

Moreover, for any $\varepsilon>0$ the inequality holds:

$$|\Phi_n'(z)| \le M(1+\varepsilon)^{n-1}\rho^{n-1}, \qquad z\in B_{1,\infty}\cup K \quad (n=1,2,...), \tag{8}$$

where M depends only on ε.

3. Expansion in a series with respect to the Taylor system.

With the aid of the properties of the functions of the Laurent system listed above, the domain of uniform convergence of the series $\sum_{n=1}^{\infty}\lambda_n\varphi_n(z)$, $\sum_{n=1}^{\infty}\lambda_n\varphi_n'(z)$ is found, and the following is established.

Theorem 2. *Every function $f(z)$, regular in the domain B_r, $1\le r<\infty$, with Taylor expansion about $z=\infty$ of the form*

$$f(z) = \sum_{k=1}^{\infty} d_k z^{-k} \tag{9}$$

can be expanded in a series with respect to the Taylor system of functions

$$f(z) = \sum_{n=1}^{\infty}\lambda_n\varphi_n(z), \tag{10}$$

converging absolutely and uniformly inside the domain B_r. Such an expansion is unique, and the expansion coefficients λ_n $(n=1,2,...)$ are determined by the formula

$$\lambda_n = \frac{i}{2\pi}\oint_{K_\rho} f(z)\,d\Phi_n(z) \quad (n=1,2,...), \qquad r<\rho<\infty, \tag{11}$$

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and satisfy the inequality

$$\overline{\lim_{n\to\infty}} |\lambda_n|^{1/n} \le r. \tag{12}$$

The quantities λ_n depend linearly on $d_1, d_2, \dots, d_n$ $(n = 1, 2, \dots)$. If the function $f(z)$ is regular in the domain B, then the area of the image B (finite or infinite) is computed by the formula

$$\sigma(f) = \iint\limits_B |f'(z)|^2 \, d\sigma = \pi \sum_{n=1}^{\infty} |\lambda_n|^2. \tag{13}$$

4. Expansion in a Laurent-type series. Let the function $\psi(z)$ be regular in some boundary annular set $B_{1,R}$ $(R > 1)$. Put

$$\sigma^*(\psi) = -\frac{i}{2} \oint\limits_K \psi(z)\overline{\psi'(z)} \, dz = \lim_{\rho\to 1} -\frac{i}{2} \oint\limits_{K_\rho} \psi(z)\overline{\psi'(z)} \, dz. \tag{14}$$

The above-mentioned properties of the Laurent system of functions make it possible to find the domain of uniform convergence of the Laurent-type series

$$\sum_{n=1}^{\infty} \Lambda_n \Phi_n'(z) + \sum_{n=1}^{\infty} \lambda_n \varphi_n'(z)$$

and to prove the following theorem.

Theorem 3. The derivative of any function $\psi(z)$, regular in the annular set $B_{r,R}$, $1 \le r < R$, can be expanded in a Laurent-type series

$$\psi'(z) = \sum_{n=1}^{\infty} \Lambda_n \Phi_n'(z) + \sum_{n=1}^{\infty} \lambda_n \varphi_n'(z), \tag{15}$$

converging absolutely and uniformly inside $B_{r,R}$.

Moreover, such an expansion is unique, and the coefficients of the expansion Λ_n and λ_n $(n = 1, 2, \dots)$ are determined by the formulas

$$\Lambda_n = -\frac{i}{2\pi} \oint\limits_{K_\rho} \psi(z) \, d\overline{\varphi_n(z)} \quad (n = 1, 2, \dots), \qquad r < \rho < R; \tag{16}$$

$$\lambda_n = \frac{i}{2\pi} \oint\limits_{K_\rho} \psi(z) \, d\Phi_n(z) \quad (n = 1, 2, \dots), \qquad r < \rho < R, \tag{17}$$

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and satisfy the inequalities

$$\overline{\lim_{n\to\infty}}\,|\Lambda_n|^{1/n} \le \frac{1}{R}, \qquad \overline{\lim_{n\to\infty}}\,|\lambda_n|^{1/n} \le r. \tag{18}$$

If the function $\psi(z)$ is regular in the boundary annular set $B_{1,R}$ $(R > 1)$, then

$$\sigma^*(\psi) = \pi\left(\sum_{n=1}^{\infty}|\Lambda_n|^2 - \sum_{n=1}^{\infty}|\lambda_n|^2\right). \tag{19}$$

II. **The system of functions $\{C_n(z)\}$.**

1. **Orthonormality of the system $\{C_n(z)\}$.** It is known that for any prescribed $t \in B$ $(t \ne \infty)$ and $\theta \in [0, \pi)$ there exists a unique function $j_\theta(z,t) \in \Sigma(B)$, $j_\theta(t,t) = 0$, which maps the domain B onto the whole plane with slits along arcs of logarithmic spirals of inclination θ to the radius vector.

Fix an arbitrary $t \in B$ and form the functions

$$P(z,t) = \frac{1}{2}\ln\frac{j_{\pi/2}(z,t)}{j_0(z,t)}, \qquad R(z,t) = \frac{1}{2}\ln\frac{(z-t)^2}{j_{\pi/2}(z,t)j_0(z,t)}, \quad z,t \in B. \tag{20}$$

$R(z,t)$ is a symmetric function of z and t in the domain B, and, for fixed

for each $t \in B$ is regular in $\overline{B}$. $P(z,t)$, for fixed $t \in B$, is regular in $\overline{B}$, and its expansion in a series with respect to the Taylor system has the form

$$P(z,t) = \sum_{n=1}^{\infty}\overline{\varphi_n(t)}\,\varphi_n(z), \qquad z,t \in B. \tag{21}$$

Take an arbitrary function $F(z) \in \Sigma(B)$ and an arbitrary $t \in B$, and construct the function of z

$$g(z,t) = \ln\frac{z-t}{F(z)-F(t)} - R(z,t), \qquad z,t \in B. \tag{22}$$

The function $g(z,t)$ is regular in the domain B and, by Theorem 2, expands in a series with respect to the Taylor system of functions for the domain B:

$$g(z,t) = \sum_{n=1}^{\infty}C_n(t)\varphi_n(z), \qquad z,t \in B. \tag{23}$$

The expansion (23), for $z \in B$, determines unique functions $C_n(z)$ $(n = 1, 2, ...)$, which are regular in the domain B. With the aid of a lemma generalizing to

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finitely connected domains a lemma previously proved by the author ([4]), the following theorems are established.

Theorem 4. For each system $\{C_n(z)\}$ $(n = 1, 2, ...)$, generated by a function $F(z) \in \widetilde{\Sigma}(B)$, the system of derivatives $\{C_n'(z)\}$ is a complete orthonormal system of functions in the domain B.

Theorem 5. For any system $\{C_n(z)\}$, generated by a function $F(z) \in \Sigma(B)$, the inequalities hold

$$\sum_{n=1}^{\infty} |C_n(z)|^2 \leq P(z, z), \qquad \sum_{n=1}^{\infty} |C_n'(z)|^2 \leq K(z, \overline{z}), \qquad z \in B, \tag{24}$$

where $K(z, \bar{t})$ is the Bergman kernel-function ([2]) for the domain B in the class $l^2(B)$, and, more generally, the inequality

$$\sum_{n=1}^{\infty} |C_n^{(\nu)}(z)|^2 \leq \sum_{n=1}^{\infty} |\varphi_n^{(\nu)}(z)|^2, \qquad z \in B \ (\nu = 0, 1, ...). \tag{25}$$

The equality sign occurs if and only if $F(z) \in \widetilde{\Sigma}(B)$.

2. Extremal systems $\{C_n(z)\}$. Using a known result of Schiffer ([5]) on Faber polynomials, one can prove the following theorem.

Theorem 6. For any sequence of complex numbers $\{x_n\}$ $(n = 1, 2, ...)$, for which

$$0 < \sum_{n=1}^{\infty} |x_n|^2 < \infty,$$

there exists at least one function $F(z)$ of the class $\Sigma(B)$ such that the system of functions $\{C_n(z)\}$ generated by it, for all $z \in B$, satisfies the equality

$$\sum_{n=1}^{\infty} x_n C_n(z) \equiv \sum_{n=1}^{\infty} \overline{x_n} \varphi_n(z), \qquad z \in B, \tag{26}$$

where $\varphi_n(z)$ $(n = 1, 2, ...)$ are the functions of the Taylor system.

From Theorems 4, 5, and 6 follow both previously known and some new results for functions of the class $\Sigma(B)$ concerning distortion theorems and conditions of univalence.

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Note: Figure translations are in progress. See original paper for figures.

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