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Abstract

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THE RIEMANN-HILBERT PROBLEM FOR THE EXTERIOR OF CUTS ALONG A STRAIGHT LINE OR ALONG A CIRCLE

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A closed-form solution is considered for the Riemann–Hilbert boundary-value problem with discontinuous coefficients for the exterior of cuts along a straight line or along a circle.

Effective solutions of boundary-value problems of potential theory have been sought by many investigators. A closed-form solution of one particular case of the Riemann–Hilbert problem with discontinuous coefficients for a circle was given by S. L. Sobolev ([1]). M. V. Keldysh and L. I. Sedov ([2]) obtained a closed-form solution of the mixed problem for a half-plane and of the Dirichlet problem for the exterior of cuts along a straight line. D. I. Sherman ([3]) solved the problem of determining an analytic function in the exterior of cuts along a straight line from prescribed values of the real part of this function on one side of the cuts and of the imaginary part on the other side. An effective closed-form solution of the Riemann–Hilbert problem with discontinuous coefficients for a disk and a half-plane was given by N. I. Muskhelishvili ([4]) and F. D. Gakhov ([5]).

1°. Let the plane of the complex variable z be cut along the segments $L_j = a_j b_j$ $(j = 1, \dots, n)$ of the real axis, having no common points; it is assumed that $a_1 < b_1 < a_2 < \cdots$. The collection of these segments will be denoted by $L = L_1 + \cdots + L_n$. It is required to determine a single-valued piecewise-analytic function $\varphi_1(z)$, with line of jumps L, in the class of integrable functions, whose boundary values on the contour L satisfy the conditions

$$\begin{aligned} &\operatorname{Re}\{[a^+(t) + ib^+(t)]\varphi_1^+(t)\} = f^+(t), \quad (t \in L), \\ &\operatorname{Re}\{[a^-(t) + ib^-(t)]\varphi_1^-(t)\} = f^-(t), \quad (t \in L). \end{aligned} \tag{1}$$

Here $a^\pm(t)$, $b^\pm(t)$, $f^\pm(t)$ are real functions, continuous almost everywhere on L, satisfying the Hölder condition on intervals of continuity (the plus and minus signs correspond respectively to the upper and lower sides of the cuts). We regard the function $\varphi_1(z)$ as bounded at infinity.

2°. We rewrite the Riemann–Hilbert problem (1) in the form

$$
\begin{aligned}
&[a^+(t) + ib^+(t)]\varphi_1^+(t) + [a^+(t) - ib^+(t)]\overline{\varphi_1^+(t)} = 2f^+(t), \\
&[a^-(t) + ib^-(t)]\varphi_1^-(t) + [a^-(t) - ib^-(t)]\overline{\varphi_1^-(t)} = 2f^-(t).
\end{aligned} \tag{2}
$$

Introduce the piecewise-analytic function $\varphi_2(z)$

$$
\varphi_2(z) = \overline{\varphi_1(\bar{z})}. \tag{3}
$$

The function $\varphi_2(z)$ is analytic everywhere in the exterior of the cuts L. The boundary condition (2), with the aid of the function $\varphi_2(z)$, is evidently reduced to the following Riemann boundary-value problem for two functions:

$$
\begin{aligned}
&\varphi_1^+(t) = \alpha(t)\varphi_2^-(t) + f_1(t), \quad (t \in L), \\
&\varphi_2^+(t) = \beta(t)\varphi_1^-(t) + f_2(t), \quad (t \in L).
\end{aligned} \tag{4}
$$

Here

$$
\begin{aligned}
&\alpha(t) = -\frac{a^+(t) - ib^+(t)}{a^+(t) + ib^+(t)}, \qquad f_1(t) = \frac{2f^+(t)}{a^+(t) + ib^+(t)}, \\
&\beta(t) = -\frac{a^-(t) + ib^-(t)}{a^-(t) - ib^-(t)}, \qquad f_2(t) = \frac{2f^-(t)}{a^-(t) - ib^-(t)}.
\end{aligned} \tag{5}
$$

The solution of the boundary-value problem (4) in closed form was obtained in the author’ s work ([6]) for an arbitrary open smooth contour L.

3°. In the case under consideration, the canonical solution ([6]) of the boundary-value problem (4) must be subject to the requirement

$$
X_2(z) = \overline{X_1}(z). \tag{6}
$$

Then the condition

$$
\nu_1 = \nu_2 = \nu, \qquad d_i = \bar{c}_i \quad (i = 1, \dots, \nu) \tag{7}
$$

will be satisfied.

The system of $(n-1)$ equations serving to determine the constants c_i and ν will be written in the form

$$
\int_L t^k \ln\left[\frac{\alpha(t)}{\beta(t)} \prod_{i=1}^{\nu} \frac{(t-\bar{c}_i)^2}{(t-c_i)^2}\right] \frac{dt}{B_n^+(t)} = 0 \qquad (k = 0, 1, \dots, n-2). \tag{8}
$$

Here $B_n^+(t)$ denotes the limiting value on the left bank of the cuts of the function, analytic outside L,

$$B_n(z) = \prod_{j=1}^{n} (z - a_j)^{1/2} (z - b_j)^{1/2}. \tag{9}$$

The canonical solution of the boundary-value problem (4) will have the form ([6])

$$X_1(z) = \prod_{i=1}^{\nu} (z - c_i) \prod_{j=1}^{n} (z - t_j)^{-\chi_j} \exp\left\{\frac{1}{2}\left[\Gamma_0(z) + \Gamma(z)\right]\right\},$$

$$X_2(z) = \prod_{i=1}^{\nu} (z - \bar{c}_i) \prod_{j=1}^{n} (z - t_j)^{-\chi_j} \exp\left\{\frac{1}{2}\left[\Gamma_0(z) - \Gamma(z)\right]\right\},$$

$$\Gamma_0(z) = \frac{1}{2\pi i} \int_L \ln[\alpha(t)\beta(t)] \frac{dt}{t - z},$$

$$\Gamma(z) = \frac{B_n(z)}{2\pi i} \int_L \ln\left[\frac{\alpha(t)}{\beta(t)} \prod_{i=1}^{\nu} \left(\frac{t - \bar{c}_i}{t - c_i}\right)^2\right] \frac{dt}{B_n^+(t)(t - z)}. \tag{10}$$

Here the arguments of the functions $\alpha(t)$ and $\beta(t)$ on each segment L_j ($j = 1, \dots, n$) are chosen in such a way that, under continuous variation of the arguments from the points $z = a_j$ and $z = b_j$ to the point $z = t_j$ (chosen arbitrarily), the singularities of the functions $X_k(z)$ at the points $z = a_j$, $z = b_j$, and at the points of discontinuity of the coefficients $\alpha(t)$ and $\beta(t)$ coincide with the prescribed class of integrable functions $\varphi_j(z)$, while at the point $z = t_j$ the prescribed class is obtained by choosing the integer χ_j (entirely analogously to the linear boundary-value problem for one function ([4],[5])).

It is easy to observe that the functions $X_1(z)$ and $X_2(z)$ satisfy condition (6), if one takes into account that the function $\Gamma_0(z)$ is real and the function $\Gamma(z)$ is purely imaginary for real values of the argument.

The solution of the Riemann boundary-value problem (4) is written in the following form ([6]):

$$\varphi_k(z) = X_k(z) \Bigg\{ \frac{1}{4\pi i} \int_L \left[\frac{f_1(t)}{X_1^+(t)} + \frac{f_2(t)}{X_2^+(t)}\right] \frac{dt}{t - z} P_\lambda(z) + \sum_{j=1}^{\nu} \frac{a_{kj}}{z - z_j} - (-1)^k \frac{B_n(z)}{4\pi i} \left[\int_L \left[\frac{f_1(t)}{X_1^+(t)} - \frac{f_2(t)}{X_2^+(t)} + 2\sum_{j=1}^{\nu} \left(\frac{a_{2j}}{t - \bar{c}_j} - \frac{a_{1j}}{t - c_j}\right)\right] \frac{dt}{B_n^+(t)(t - z)} + P_\mu(z)\right] \Bigg\}, \tag{11}$$

where $\chi = \sum_{j=1}^{n} \chi_j$; ($z_i = c_i$ for $k = 1$ and $z_i = \bar{c}_i$ for $k = 2$); $a_{2j} = \bar{a}_{1j}$.

$P_n(z)$ is a polynomial of degree n; generally speaking, $\nu = n - 1$; $a_{jk} = \lim_{z \to z_k} (z - z_k)\varphi_j(z) X_j^{-1}(z)$.

For $\nu - \chi + n - 1 > 0$, $\mu = 0, \lambda = n - 1$.

For $\nu - \chi + n - 1 < 0$, $\mu = \chi - n + 1 - \nu, \lambda = \chi - \nu$.

In the case when the solution (11) is unbounded at infinity, the corresponding solvability conditions must also be satisfied.

It can be shown that the functions $\varphi_1(z)$ and $\varphi_2(z)$ satisfy conditions (3) if the coefficients of the polynomial are real.

It should be noted that the detailed analysis, carried out by M. V. Keldysh, L. I. Sedov, and N. I. Muskhelishvili, of the solvability of the Dirichlet problem for the exterior of cuts along a straight line in various classes of functions is, in the Riemann-Hilbert problem under consideration, made difficult by the necessity of solving the system of nonlinear equations (8).

In formula (11) it may happen that the product $[B_n(z) X_{1,2}(z)]^{-1}$ has a non-integrable singularity at some endpoints of the segments L_j, $z = g_i$, so that formula (11), generally speaking, loses its meaning. In this case, instead of $B_n(z)$ in (11), as is easy to see, one should take the function $B_n(z) \prod_i (z - g_i)^{-1}$. Accordingly, formulas (11) will be written somewhat differently. In the case when the contour L contains the point at infinity, the integrals in formulas (10), (11), and (12) may turn out to be divergent. In this connection one may modify the definition of the canonical function of work ([6]), taking as the distinguished point some finite point of the z-plane not lying on the contour L. One may also, by a fractional-linear transformation, reduce the original Riemann-Hilbert problem to the one already considered. Finally, the divergent integrals may be conventionally understood in a certain generalized sense. An analogous remark is also valid with respect to formulas (1.9) and (1.10) of work ([6]).

In solving concrete problems it is more convenient to apply directly the solution procedure indicated in work ([6]) and in the present note.

We note that, for all the particular cases considered earlier and indicated in the introductory part, the system of equations (8) can be satisfied by putting $\nu = 0$ ([6]).

4°. Let the plane be cut along a finite number of arcs of some circle, on which the boundary condition of the Riemann-Hilbert problem is prescribed. The solution of this problem is completely analogous to the one just considered.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.