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Abstract

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BOUNDARY-VALUE PROBLEMS FOR OPERATORS IN TWO VARIABLES WITH NONHOMOGENEOUS PRINCIPAL PART

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For operators in two variables t, x, considered in a bounded convex domain with piecewise smooth boundary, boundary-value problems have been well studied in the case of a homogeneous principal part ([8]) (in particular, for hyperbolic ([7]) and elliptic equations), and also for certain special classes of operators that are, in a certain sense, the most direct generalization of the parabolic and elliptic cases ([2,3,5]). In the cited works, boundary-value problems more general than those considered by us were studied.

In the present work we consider the simplest boundary-value problems for operators with nonhomogeneous principal part. Restricting ourselves for the time being to the case where the principal part is a binomial, we show that, depending on the location of the roots of the characteristic equation, the operators under consideration split into five types. For each type its own class of boundary-value problems is indicated. The method of investigation applied makes it possible to show that these classes of problems describe, in a certain sense, all admissible boundary conditions for each type of operators. This is established by constructing examples showing the ill-posedness or instability of problems different from those considered. The approach used is close to ([1]).

We pass to the statement of the results.

Consider, in the domain $V = (0 \le t \le T,\ 0 \le x \le 2l)$, the equation

$$Lu + L_1(t,x)u = f, \tag{1}$$

where

$$Lu \equiv \frac{\partial^m u}{\partial t^m} + \frac{\partial^n u}{\partial x^n}; \qquad L_1(t,x)u \equiv \sum_{r=r_1/m+r_2/n_0<1} a_r(t,x)\frac{\partial^r u}{\partial t^{r_1}\partial x^{r_2}};$$

n_0 is some positive integer, $n_0 < n$, determined by the type of the operator L; $a_{m-1}(t,x) \in C^{m-1,n}_{t,x}$ and is periodic in x with period $2l$; $a_r(t,x)$, $r_1 \neq m-1$, are summable functions bounded on V, $f \in \mathcal{L}_2(V)$.

We shall be interested in the character of the solvability of equation (1) under boundary conditions of the form

$$u_{t_i}^{(r_i)}(T,x)=0, \qquad i=1,2,\dots,\nu;$$

$$u_{t_j}^{(s_j)}(0,x)=0, \qquad j=1,2,\dots,m-\nu, \tag{Γ}$$

where r_i, s_j are integers and $0\le r_1<r_2<\dots<r_\nu<m,\ 0\le s_1<s_2<\dots<s_{m-\nu}<m$, and under periodicity conditions in x. Here ν is a positive integer that determines, for each type of operator L, its own class of problems.

Consider the equation determined by the operator L,

$$\omega^m+(i\xi)^n=0. \tag{2}$$

For real $\xi\ne 0$, among the roots $\omega_j,\ i=1,2,\dots,m$, of equation (2) there are no multiple roots. The possible types of location of the roots ω_j do not depend on the values $\xi\ne 0$ and determine the following five distinct types of the operator L:

I. $m=2m_1$, and equation (2) has m_1 roots ω_j with $\operatorname{Re}\omega_j>0$ and m_1 roots ω_j with $\operatorname{Re}\omega_j<0$. Then $\nu=m_1$ (Dirichlet problem).

II. $m=2m_1$ and equation (2) has (m_1-1) roots ω_j with $\operatorname{Re}\omega_j>0$ and (m_1-1) roots ω_j with $\operatorname{Re}\omega_j<0$. Moreover, two roots are equal to $\pm i$. Then $\nu=m_1-1$ or $\nu=m_1+1$.

III. $m=2m_1+1$ and equation (2) has m_1 roots ω_j with $\operatorname{Re}\omega_j>0$, m_1 roots ω_j with $\operatorname{Re}\omega_j<0$, and one purely imaginary root. Then $\nu=m_1$ or $\nu=m_1+1$.

IV. $m=2m_1+1$ and equation (2) has m_1 roots ω_j with $\operatorname{Re}\omega_j>0$ and (m_1+1) roots ω_j with $\operatorname{Re}\omega_j<0$. Then $\nu=m_1$.

V. $m=2m_1+1$ and equation (2) has (m_1+1) roots ω_j with $\operatorname{Re}\omega_j>0$ and m_1 roots ω_j with $\operatorname{Re}\omega_j<0$. Then $\nu=m_1+1$.

In what follows we shall speak of the operator L and of the boundary conditions (Γ), understanding by L any one of the listed types I-V and associating with each type its own number of boundary conditions ν in (Γ), defined above.

Denote by $P_{t,x}^{m+1,n+1}$ the set of complex functions having continuous periodic partial derivatives of period $2l$ in t up to order $m+1$ and in x up to order $n+1$. Let, for $u\in P_{t,x}^{m+1,n+1}$,

$$u=\sum_{k=-\infty}^{+\infty}u_k(t)e^{ik\pi x/l}, \qquad Lu\equiv\varphi=\sum_{k=-\infty}^{+\infty}\varphi_k(t)e^{ik\pi x/l}.$$

If $u \in P^{m+1,n+1}_{t,x}$ satisfies (Γ) and the equation

$$\frac{\partial^m u}{\partial t^m} + \frac{\partial^n u}{\partial x^n} = \varphi, \tag{3}$$

then $u_k(t)$ satisfies the conditions

$$\frac{d^m u_k}{dt^m} + \left(\frac{ik\pi}{l}\right)^n u_k = \varphi_k; \tag{4}$$

$$u_k^{(r_i)}(T) = 0, \quad i = 1, 2, \dots, \nu; \qquad u_k^{(s_j)}(0) = 0, \quad j = 1, 2, \dots, m - \nu, \tag{5}$$

$$0 \le r_1 < r_2 < \dots < r_\nu < m; \quad 0 \le s_1 < s_2 < \dots < s_{m-\nu} < m.$$

Lemma 1. For each of the operators of types I–V, for all k the following inequalities hold:

I. $$|k|^r |u_k(t)| \le C_1 |k|^{-(n - n/4m_1 - r)} \|\varphi_k\|_t, \qquad r = 0, 1, \dots, n_0 = \left[n - \frac{n}{4m_1}\right].$$

II. $$|k|^r |u_k(t)| \le C_2 |k|^{-(n - n/2m_1 - r)} \|\varphi_k\|_t, \qquad r = 0, 1, \dots, n_0 = \left[n - \frac{n}{2m_1}\right].$$

III. $$|k|^r |u_k(t)| \le C_3 |k|^{-(n - n/(2m_1+1) - r)} \|\varphi_k\|_t, \qquad r = 0, 1, \dots, n_0 = \left[n - \frac{n}{2m_1 + 1}\right].$$

IV. $$|k|^r |u_k(t)| \le C_4 |k|^{-(n - n/2(2m_1+1) - r)} \|\varphi_k\|_t,$$

$$r = 0, 1, \dots, n_0 = \left[n - \frac{n}{2(2m_1 + 1)}\right].$$

V. $$|k|^r |u_k(t)| \le C_5 |k|^{-(n - n/2(2m_1+1) - r)} \|\varphi_k\|_t,$$

$$r = 0, 1, \dots, n_0 = \left[n - \frac{n}{2(2m_1 + 1)}\right].$$

Remark. In essence, in each of the cases one should write a pair of inequalities. For example, in case I, in addition to the displayed inequality, an inequality of the form

$$|u_k^{(r)}(t)| \le C_1' \alpha_k^{-(2m_1-1/2-r)} \|\varphi_k\|_t, \qquad \alpha_k = |k|^{n/m}, \quad r = 0, 1, \dots, 2m_1 - 1$$

is valid.

In the estimates given, C does not depend on k, the square brackets denote the integer part, and

$$\|\varphi_k\|_t^2 = \int_0^T |\varphi_k(t)|^2 \, dt.$$

For $k = 0$ the validity of the estimates of Lemma 1 is obvious (in some cases, it may be necessary to impose additional conditions in order to exclude polynomials in t from consideration). For $k \ne 0$ the estimates of Lemma 1 are obtained from the formula for the solution of problem (4), (5):

$$\begin{aligned} u_k(t) = &-\frac{(-1)^m}{\alpha_k^{m-1} W_m(\omega)} \sum_{s=1}^m e^{\alpha_k \omega_s t} \Bigg[(-1)^s W_{m-1}(\omega \ne \omega_s) \int_0^t e^{-\alpha_k \omega_s \tau} \varphi_k \, d\tau \\ &- \frac{1}{\Delta(\alpha_{kT})} \sum_{\substack{p'=1 \\ p' \ne s}}^m (-1)^{p'_{p'}} D_{m-1}(\omega \ne \omega_{p'}) \\ &\times \sum_{q=1}^m (-1)^q W_{m-1}(\omega \ne \omega_q) D_\nu(\omega_q, \omega_{p'}) e^{\alpha_{kT}(\omega_q + |\omega_{p'}|)} \int_0^T e^{-\alpha_k \omega_q \tau} \varphi_k \, d\tau \Bigg], \end{aligned} \tag{6}$$

where $\alpha_k = |k|^{n/m}$; $\alpha_k \omega_1, \dots, \alpha_k \omega_m$ are the roots of the characteristic equation for (4); $W_m(\omega)$ is the Vandermonde determinant of order m formed from the numbers $\omega = (\omega_1, \dots, \omega_m)$, and $W_{m-1}(\omega \ne \omega_s)$ from the numbers $\omega_1, \dots, \omega_{s-1}, \omega_{s+1}, \dots, \omega_m$; $p = (p', p_\nu)$; $p' = (p_1, \dots, p_{\nu-1})$; $\omega_p = (\omega_{p'}, \omega_{p_\nu})$; $\omega_{p'} = (\omega_{p_1}, \dots, \omega_{p_{\nu-1}})$; $|p'| = p_1 + \dots + p_{\nu-1}$; $|p| = |p'| + p_\nu$; $|\omega_{p'}| = \omega_{p_1} + \dots + \omega_{p_{\nu-1}}$; $\omega_p = |\omega_{p'}| + \omega_{p_\nu}$; p_i, p_j, and q are pairwise distinct;

$$p_p = m\nu - \frac{1}{2}\nu(\nu - 1) + |p|;$$

$$p'_{p'} = m\nu - \frac{1}{2}\nu(\nu-1) + |p'| + s; \qquad \Delta(\alpha_{kT}) = \sum_{p=1}^m (-1)^{p_p} D_\nu(\omega_p) D_{m-\nu}(\omega \ne \omega_p) e^{\alpha_{kT}|\omega_p|},$$

where D_ν and $D_{m-\nu}$ are determinants of order ν and $m-\nu$, respectively, for example

$$D_\nu(\omega_p) = \begin{vmatrix} \omega_{p_1}^{r_1} & \cdots & \omega_{p_\nu}^{r_1} \\ \cdots & \cdots & \cdots \\ \omega_{p_1}^{r_\nu} & \cdots & \omega_{p_\nu}^{r_\nu} \end{vmatrix}.$$

Following (4) (p. 78), one can show that $\Delta(\alpha_{kT}) \neq 0$.

Putting, as usual,

$$(u,v) = \int_V u\bar{v}\,dV; \qquad |u,H|^2 = (u,u)$$

and completing $P_{t,x}^{m+1,n+1}$ with respect to the introduced norm, we obtain a Hilbert space H of square-summable functions.

It follows from Lemma 1 that on the set $u \in P_{t,x}^{m+1,n+1}$ satisfying (3), (Γ), one can introduce the scalar product

$$\langle u,v\rangle = (Lu, Lv). \tag{7}$$

The Hilbert space obtained by completing $P_{t,x}^{m+1,n+1}$ in the metric (8) will be denoted by $\mathscr{H}$. The inequalities of Lemma 1 show that the embedding

$$\mathscr{H} \subseteq H_{t,x}^{m-1,n_0},$$

is valid, where $H_{t,x}^{m-1,n_0}$ is the Hilbert space of functions having generalized partial derivatives with respect to t of order $(m-1)$ and with respect to x of order n_0.

A function $u \in \mathscr{H}$ will be called a **strong solution** of problem (1), (Γ), if there exists a sequence $\{u_i\}$ of functions from $P_{t,x}^{m+1,n+1}$, satisfying the conditions (Γ), for which

$$|u_i - u, \mathscr{H}| \to 0, \qquad |Lu_i + L_1(t,x)u_i - f, H| \to 0 \quad \text{as } i \to \infty.$$

Lemma 2. For every function $\varphi \in H$ there exists a unique strong solution of problem (3), (Γ).

The validity of this lemma follows from Lemma 1 and from the possibility of approximating $\varphi \in H$ by finite sums of the form $\sum \varphi_k e^{ik\pi x/l}$ with smooth functions φ_k.

Theorem 1. Problem (1), (Γ) is regularly solvable, i.e., the three Fredholm theorems are valid for it.

Remark. After the substitution

$$u(t,x) = v(t,x) \exp\left[-\frac{1}{m}\int_0^t a_{m-1}(\tau,x)\,d\tau\right]$$

for the function $v(t,x)$ one obtains an equation of the same form as (1), but with $a_{m-1}(t,x) = 0$. Therefore it is sufficient to carry out the proof of Theorem 1 for the equation

$$Lu + L_2(t,x)u = f, \tag{8}$$

where for $L_2(t,x)$

$$r = \frac{r_1}{m-1} + \frac{r_2}{n_0} < 1.$$

From Lemma 2 there follows the existence of the operator $L^{-1} : H \to \mathcal{H}$, and hence also the existence of the operator $A = L^{-1}L_2 : \mathcal{H} \to \mathcal{H}$. In this case problem (9), (Γ) will be equivalent to the operator equation in the space $\mathcal{H}$

$$u + Au = g \equiv L^{-1}f. \tag{9}$$

Using the inequalities of Lemma 1, the embedding theorems ([6]) for $m > 1$, $n > 1$, and proceeding as in ([5]) for $m = 1$ or $n = 1$, one can show that A is a completely continuous operator, i.e., equation (9) is of Fredholm type, which proves Theorem 1.

Theorem 2. For sufficiently small T, for every function $f \in H$ there exists a unique strong solution of (1), (Γ).

We indicate in conclusion a method of constructing examples showing that in each case going outside the described class of problems leads to ill-posedness or instability.

Example. Let

$$Lu \equiv \frac{\partial^4 u}{\partial t^4} + \frac{\partial^4 u}{\partial x^4} = e^{ik\pi x/l} \equiv f^{(k)}.$$

For this equation, which is elliptic, the Dirichlet problem is known to be well posed. Let us now impose on the function $u(t,x)$ the conditions

$$u(T,x) = u(0,x) = u_t(0,x) = u_{t^2}(0,x) = 0$$

and the periodicity condition in x.

The solution of this problem has the form

$$u_k(t) = \frac{l^4}{\pi^4 k^4} e^{ik\pi x/l}\left[1 - \operatorname{ch}\alpha_k t \cos\alpha_k t - \frac{1 - \operatorname{ch}\alpha_k T\cos\alpha_k T}{\Delta(\alpha_{kT})}\Delta(\alpha_k t)\right],$$

where

$$\alpha_k = \frac{\pi k}{l\sqrt{2}}, \qquad \Delta(\alpha_k t) = \operatorname{ch}\alpha_k t \sin\alpha_k t - \operatorname{sh}\alpha_k t\cos\alpha_k t, \qquad \Delta(\alpha_{kT}) = \Delta(\alpha_k t)\big|_{t=T}.$$

For $k \to \infty$

$$k^4 \Delta(\alpha_{kT}) = O\left[k^4 e^{\alpha_{kT}} \cos(\alpha_{kT} - 3\pi/4)\right].$$

From the formula for the solution we obtain that, for $0 < t = t_0 < T,\ k \to \infty$, and

$$\alpha_{kT} - 3\pi/4 \neq \pi/2 + n\pi, \qquad n = 0, 1, 2, \dots,$$

$|u^{(k)}| \to \infty$, whereas $|f^{(k)}| = 1$. On the other hand, by changing T and l one can obtain

$$\alpha_{kT} - 3\pi/4 = \pi/2 + n\pi,$$

and then, as is not hard to show, the homogeneous problem has a nontrivial solution, i.e., uniqueness is violated. Thus the chosen problem possesses ill-posedness similar to the Cauchy problem for the Laplace operator and at the same time instability analogous to the instability of the Dirichlet problem for the wave equation.

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Note: Figure translations are in progress. See original paper for figures.