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Abstract

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ON THE STABILITY OF MOTION IN THE DOUBTFUL CASE OF TWO ZERO ROOTS

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In paper ([1]), Lyapunov studied the question of stability in the doubtful case of two zero roots with a nonsimple elementary divisor, when the number of equations of the system is greater than two. In this connection, only in one case was Lyapunov' s analysis not exhaustive. In this case the problem reduces to the study of the following system of differential equations:

$$\begin{aligned} \frac{dr}{dt} &= r^{q+1}R_1(r,\vartheta) + rR_2(r,z,\vartheta), \\ \frac{d\vartheta}{dt} &= r^{q-1} + r^q\Theta_1(r,\vartheta) + \Theta_2(r,z,\vartheta), \\ \frac{dz}{dt} &= Az + Z(r,z,\vartheta), \end{aligned} \tag{1}$$

where q is a natural number greater than one; r and ϑ are scalar variables; z is an n-dimensional vector with components $z_1, \dots, z_n$; $A = \{a_{ij}\}$ $(i, j = 1, \dots, n)$ is a constant square matrix with negative real parts of the characteristic roots. The functions R_1 and Θ_1 are power series in r with coefficients that are ω-periodic with respect to ϑ, converging absolutely and uniformly for sufficiently small r and all ϑ. The functions R_2 and Θ_2 are analogous series in powers of $r, z_1, \dots, z_n$, with R_2 and Θ_2 vanishing when $z = 0$. The components Z_s of the vector-function Z are also series in powers of $r, z_1, \dots, z_n$ with coefficients ω-periodic in ϑ; these series contain no terms of dimension lower than the second in r, z_s, and r enters terms independent of z_s only in powers not lower than $2q$. In this case one may assume that $r \geq 0$.

To study system (1), Lyapunov introduces new variables by the following formulas:

$$r = c(1+\rho), \qquad z = c^\beta \zeta, \tag{2}$$

where c is a sufficiently small positive constant, and β is a natural number satisfying the inequalities

$$q \leq \beta < 2q - 1. \tag{3}$$

Substituting (2) into system (1) and eliminating t, we obtain

$$\begin{aligned} \frac{d\rho}{d\vartheta} &= cP(c, \rho, \zeta, \vartheta), \\ c^{q-1}\frac{d\zeta}{d\vartheta} &= A\zeta + cF(c, \rho, \zeta, \vartheta), \end{aligned} \tag{4}$$

where P is a scalar and F a vector function, expandable in series in powers of c, ρ, ζ_s with coefficients ω-periodic in ϑ; these series converge absolutely and uniformly for all ϑ and sufficiently small $c, |\rho|$, and $\|\zeta\|$ (where $\|\zeta\|$ denotes the Euclidean norm of the vector ζ).

Along with system (4), Lyapunov considered the system

$$\begin{aligned} \frac{d\rho}{d\vartheta} &= CP(c, \rho, \xi, \vartheta) + \varphi(c), \\ c^{q-1}\frac{d\xi}{d\vartheta} &= A\xi + cF(c, \rho, \xi, \vartheta), \end{aligned} \tag{5}$$

where the function $\varphi(c)$, defined for sufficiently small nonnegative c, is chosen in such a way that system (5), for all sufficiently small $c \geqslant 0$, has an ω-periodic solution $\rho = \rho_0(c, \vartheta)$, $\xi = \xi_0(c, \vartheta)$ with the following properties:

$$\rho_0(c, 0) = 0, \qquad |\rho_0(c, \vartheta)| < Dc, \qquad \|\xi_0(c, \vartheta)\| < Dc^{2q-\beta}, \tag{6}$$

where D is a certain positive constant.

Lyapunov proved the existence of the function $\varphi(c)$ and investigated its properties in detail; in particular, he established that the function $\varphi(c)$ is continuously differentiable.

From the form of systems (4) and (5) it follows that for those c for which $\varphi(c) = 0$, system (4) has an ω-periodic solution $\rho = \rho_0(c, \vartheta)$, $\xi = \xi_0(c, \vartheta)$ with properties (6).

In paper [1], the only case left unresolved was when the function $\varphi(c)$ can take both zero and negative values in any neighborhood of the point $c = 0$. However, one can prove a statement from which it follows that in this case stability holds. Namely, the following theorem is valid.

Theorem. *If there exists a set C of positive numbers, having zero as its limit point, such that $\varphi(c) = 0$ for $c \in C$, then the zero solution of system (1) is stable with respect to the variables r and z.*

The proof of this theorem is essentially based on the following considerations. Choose a sufficiently small $c \in C$; to it there corresponds the periodic solution $\rho = \rho_0(c, \vartheta)$, $\xi = \xi_0(c, \vartheta)$ of system (4).

In system (4), make the change of variables:

$$\rho = \rho_0(c, \vartheta) + \alpha, \qquad \xi = \xi_0(c, \vartheta) + \psi. \tag{7}$$

The equations for α and ψ have the form

$$\frac{d\alpha}{d\vartheta} = cQ(c, \alpha, \psi, \vartheta), \qquad c^{q-1}\frac{d\psi}{d\vartheta} = A\psi + cG(c, \alpha, \psi, \vartheta), \tag{8}$$

where the functions Q and G satisfy the relations

$$Q(c, 0, 0, \vartheta) \equiv 0, \qquad G(c, 0, 0, \vartheta) \equiv 0; \tag{9}$$

$$\begin{aligned} |Q(c, \alpha_1, \psi_1, \vartheta) - Q(c, \alpha_2, \psi_2, \vartheta)| &< L(|\alpha_1 - \alpha_2| + \|\psi_1 - \psi_2\|), \\ \|G(c, \alpha_1, \psi_1, \vartheta) - G(c, \alpha_2, \psi_2, \vartheta)\| &< L(|\alpha_1 - \alpha_2| + \|\psi_1 - \psi_2\|). \end{aligned} \tag{10}$$

The Lipschitz conditions (10) are fulfilled for all sufficiently small $c \in C$, $|\alpha|$, $\|\psi\|$, and for all ϑ; moreover, the constant L can be chosen the same for all sufficiently small $c \in C$ (and, correspondingly, for all periodic solutions $\rho = \rho_0(c, \vartheta)$, $\xi = \xi_0(c, \vartheta)$ of system (4)).

Relying on the fact that all eigenvalues of the matrix A have negative real parts and that, in conditions (10), the Lipschitz constant L does not depend on c, one can prove the following assertion.

Lemma. *There exist constants $\lambda > 0$, $c_0 > 0$, $a_0 > 0$ and a scalar function $f(c, a, \theta)$, where a is an n-dimensional vector, such that:*

1) *the function f is defined for any $c \in C$, $c \leqslant c_0$, $\|a\| \leqslant a_0$, and for all θ;*
2) *f is continuous with respect to a and θ for any fixed c in its domain of definition;*
3)
$$f(c, a, \theta + \omega) = f(c, a, \theta);$$
4)
$$|f(c, a, \theta)| \leqslant \|a\|\, c^{q-1};$$
5) for any solution of system (8) with initial data

$$\vartheta = \theta, \qquad \psi = a, \qquad \alpha = f(c, a, \theta), \qquad \|a\| \leqslant a_0$$

the inequalities

$$\|\psi(\vartheta)\| < E\|a\| \exp\left[-\frac{\lambda}{c^{q-1}}(\vartheta - \theta)\right],$$

$$|\alpha(\vartheta)| < \|a\| c^{q-1} \exp\left[-\frac{\lambda}{c^{q-1}}(\vartheta - \theta)\right]$$

hold for $\vartheta \geqslant \theta$, where E is some positive number.

Let us indicate the method for constructing the function f. Consider the system of integral equations:

$$\alpha = -c \int_{\vartheta}^{+\infty} Q(c, \alpha, \psi, \tau)\, d\tau,$$

$$\psi = \exp\left[\frac{1}{c^{q-1}}(\vartheta - \theta)\right] a + \frac{1}{c^{q-2}} \int_{\theta}^{\vartheta} \exp\left[\frac{A}{c^{q-1}}(\vartheta - \tau)\right] G(c, \alpha, \psi, \tau)\, d\tau, \quad (11)$$

where a is an n-dimensional constant vector.

Using the fact that all eigenvalues of the matrix A have negative real parts and that the functions θ and G satisfy conditions (10), one can show, by applying the method of successive approximations with initial approximation

$$\alpha_0 = 0, \qquad \psi_0 = \exp\left[\frac{A}{c^{q-1}}(\vartheta - \theta)\right] a,$$

that system (11) has a solution $\alpha(c, a, \vartheta, \theta)$, $\psi(c, a, \vartheta, \theta)$, ω-periodic in ϑ and θ, and one must set

$$f(c, a, \theta) = \alpha(c, a, \theta, \theta).$$

The formulated lemma makes it possible to prove the following assertion. Let the solution $r(t)$, $\vartheta(t)$, $z(t)$ of system (1) have initial data $r = \bar{r}$, $\vartheta = \bar{\vartheta}$, $z = \bar{z}$ at $t = \bar{t}$, and let $\varepsilon > 0$ be an arbitrary number; then, for all $t \geqslant \bar{t}$, the relations

$$r(t) < \varepsilon, \qquad \|z(t)\| < \varepsilon$$

hold, provided only that

$$\|\bar{z}\| < \delta, \qquad \bar{r} \leqslant c\,[1 + \rho_0(c, \bar{\vartheta}) + f(c, c^{-p}\bar{z} - \zeta_0(c, \bar{\vartheta}), \bar{\vartheta})],$$

where $c \in C$ and δ are sufficiently small. This proves the assertion of the theorem.

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REFERENCES

1. A. M. Lyapunov, "Investigation of one of the special cases of the problem of stability of motion," L., 1963.

Note: Figure translations are in progress. See original paper for figures.