V. I. DISKANT

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Abstract

Full Text

V. I. DISKANT

STABILITY IN LIEBMANN' S THEOREM

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1. A stability theorem corresponding to Liebmann' s theorem on the rigidity of the sphere was proved by A. I. Fet in ([4]). In the present work an exact estimate is given for the order of the stability function $\varphi(\varepsilon)$ (see ([4])) with respect to ε. Namely, we prove

Theorem 1. *If the Gaussian curvature K of a convex n-dimensional surface Γ satisfies the inequalities $1-\varepsilon \leqslant K \leqslant 1+\varepsilon$, then for the radius r of the largest inscribed ball and for the radius R of the smallest circumscribed ball the estimates*

$$r \geqslant 1 - C\varepsilon, \qquad R \leqslant 1 + C\varepsilon,$$

hold, where C is a number depending only on n, $0 \leqslant \varepsilon \leqslant \frac{1}{2}$.

The sharpness of the estimate follows from consideration of spheres of curvature $1-\varepsilon$ and $1+\varepsilon$.

2. We first note two lemmas for a surface of revolution.

Lemma 1. *If the Gaussian curvature K of a convex surface Σ, formed by the rotation of a curve $y = y(x)$ $(y \geqslant 0)$ about the x-axis, satisfies the inequality $K \geqslant a^n$, where $a > 0$, then $\max y(x) \leqslant 1/a$.*

Proof. Let $\max y(x) = y(0)$. Suppose that $y(0) > 1/a$. Consider the curve

$$x = x_a(y) = \begin{cases} \displaystyle\int_y^{y(0)} \frac{\left[a^n(y^n - y^n(0)) + 1\right]^{1/n}}{\sqrt{1 - \left[a^n(y^n - y^n(0)) + 1\right]^{2/n}}}\, dy, & \text{if } y \neq y(0), \\ 0, & \text{if } y = y(0). \end{cases}$$

The surface of revolution of the curve $x = x_a(y)$ about the x-axis has curvature $K(x) \equiv a^n$. Applying Lemma 1 ([3]) to the curves $x = x_a(y)$ and $x = x(y)$, we obtain $x'(\bar{y}) = 0$, where $\bar{y} = \sqrt[n]{y^n(0) - 1/a^n} > 0$. The latter is impossible.

Lemma 2. *If the curve $y = y(x)$ is defined on $[0, \eta]$ and satisfies the conditions*

$$y(x) \geqslant 0, \qquad y'(x) \leqslant 0, \qquad y''(x) < 0, \qquad y'(\eta) = 0, \qquad y(0) \leqslant 1/a,$$

$$K(x) \geqslant a^n,$$

then all points of this curve belong to the figure G, whose boundary consists of the curve $x = x_a(y)$ $(0 \leqslant y \leqslant y(0))$, the segments $[0, x_a(0)]$ of the x-axis, and $[0, y(0)]$ of the y-axis.

Lemma 2 is a direct consequence of Lemma 1 ([3]).

3. The assertion of Theorem 1 concerning the radius r follows from the following estimate.

Proposition A. *If the Gaussian curvature K of a convex surface Γ satisfies the inequalities $a^n \leqslant K \leqslant b^n$, then for the radius r of the largest inscribed ball the estimate*

$$r \geqslant A + \left(B^{n+1} - A^{n+1}\right) \frac{(n-1)\chi_{n+1}}{\chi_n} \left[\frac{1}{A^n} + \frac{1}{B^n} \left(\frac{A}{B} \right)^{n^2+n+1} \right], \qquad (1)$$

holds, where χ_n is the volume of the n-dimensional unit ball, $A = 1/a$, $B = 1/b$.

The proof of Proposition A is based on the following lemma.

Lemma 3. Let the convex body T be bounded by a surface of revolution Σ about the x-axis. Suppose that the Gaussian curvature K of the surface Σ satisfies the inequality $K \geq a^n$ and that the volume of the body $V_T \geq \varkappa_{n+1} B^{n+1}$. Then, for the distance $\rho(Z, X_0)$ between the center of gravity $Z = (z, 0, \dots, 0)$ of the body T and the point of intersection $X_0 = (x(0), 0, \dots, 0)$ of the x-axis with the surface Σ, the estimate

$$\rho(Z, X_0) \geq A + (B^{n+1} - A^{n+1}) \frac{\varkappa_{n+1}(n+1)}{\varkappa_n} \left[\frac{1}{A^n} + \frac{1}{B^n} \left(\frac{A}{B} \right)^{n^2+n+1} \right]. \qquad (2)$$

Proof of Lemma 3. Let Σ be obtained by rotating the curve $y = y(x)$ $(y \geq 0)$. Suppose that $y(0) = \max y(x)$. A suitable approximation allows us to assume that $y = y(x)$ satisfies all the conditions of Lemma 1 and, for $x \geq 0$, the conditions of Lemma 2, for, by Lemma 1, $y(0) \leq A$. Consider in the xy-plane the figure M, composed of the figure G and the figure symmetric to G with respect to the y-axis.

Introduce the following notation: T_M is the body obtained by rotating the figure M about the x-axis; V_M is the volume of T_M; T_r (T_l) is the part of the body T lying in the half-space $x \geq 0$ $(x \leq 0)$; V_r (V_l) is the volume of T_r (T_l); T_r^M (T_l^M) is the part of the body T_M lying in the half-space $x \geq 0$ $(x \leq 0)$; $T_\mathrm{r}^0 = T_\mathrm{r}^M \setminus T_\mathrm{r}$, $T_\mathrm{l}^0 = T_\mathrm{l}^M \setminus T_\mathrm{l}$; V_r^0 (V_l^0) is the volume of T_r^0 (T_l^0); z_r (z_l) is the abscissa of the center of gravity of T_r (T_l); z_r^M (z_l^M) is the abscissa of the center of gravity of T_r^M (T_l^M); z_r^0 (z_l^0) is the abscissa of the center of gravity of T_r^0 (T_l^0).

Since $\rho(Z, X_0) = x(0) - z = |x(0) - z| \geq x(0) - |z|$, it suffices to find the required estimates for $x(0)$ and $|z|$.

Let us estimate $|z|$. By the additivity property of the center of gravity we have

$$\frac{V_M}{2} z_{\mathrm{r}}^M = V_{\mathrm{r}} z_{\mathrm{r}} + V_{\mathrm{r}}^0 z_{\mathrm{r}}^0 \quad \left(\frac{V_M}{2} z_{\mathrm{l}}^M = V_{\mathrm{l}} z_{\mathrm{l}} + V_{\mathrm{l}}^0 z_{\mathrm{l}}^0\right); \tag{3}$$

$$V_{Tz} = V_{\mathrm{l}} z_{\mathrm{l}} + V_{\mathrm{r}} z_{\mathrm{r}}. \tag{4}$$

From (3) and (4) we find

$$z = \frac{-V_{\mathrm{l}}^0 z_{\mathrm{l}}^0 - V_{\mathrm{r}}^0 z_{\mathrm{r}}^0}{V_T}; \tag{5}$$

from the condition of the lemma

$$V_T \geq \varkappa_{n+1} B^{n+1}. \tag{6}$$

Applying Theorem 2 (?) to the sphere of radius A and to the surface of the body T_M, we obtain

$$V_T \leq V_M \leq \varkappa_{n+1} A^{n+1}. \tag{7}$$

From (6) and (7) we have

$$V_{\mathrm{r}} \leq \frac{V_M}{2} \leq \frac{1}{2} \varkappa_{n+1} A^{n+1} \quad \left(V_{\mathrm{l}} \leq \frac{1}{2} \varkappa_{n+1} A^{n+1}\right);$$

therefore

$$V_{\mathrm{r}} = V_T - V_{\mathrm{l}} \geq \varkappa_{n+1} B^{n+1} - \frac{1}{2} \varkappa_{n+1} A^{n+1} \quad \left(V_{\mathrm{l}} \geq \varkappa_{n+1} B^{n+1} - \frac{1}{2} \varkappa_{n+1} A^{n+1}\right),$$

whence

$$V_{\mathrm{r}}^0 \leq \varkappa_{n+1} (A^{n+1} - B^{n+1}) \quad (V_{\mathrm{l}}^0 \leq \varkappa_{n+1} (A^{n+1} - B^{n+1})). \tag{8}$$

To estimate z_{r}^0 (z_{l}^0), note that $z_{\mathrm{r}}^0 \leq x_a(0)$. For $x_a(0)$ the inequality

$$\frac{1}{n+1} \varkappa_n y^n(0) x_a(0) \leq \frac{\varkappa_{n+1} A^{n+1}}{2}, \tag{9}$$

is valid, in whose left-hand side stands the volume of the cone contained in T_{r}^M. Obviously, $y(0) \geq \Delta/2$, where Δ is the width of the body T.

Replace in Theorem 3 ([3]) the estimate for V_S in formula (7) by the inequality

$$V_S \ll 2V_C \ll \varkappa_{n+1} A^n x(0). \tag{10}$$

Then in Theorem 3 ([3]) for Δ we obtain the estimate

$$\Delta \gg 2B \left(\frac{B}{A}\right)^n. \tag{11}$$

Let us prove (10). Note that V_C is the volume of a segment of a ball of radius A, $x(0)$ ($x(0) \ll A$) is the height of the segment. Denote by S the area of the base of the segment. Put $V_C = hP$ and $\frac{1}{2}\varkappa_{n+1}A^{n+1} - V_C = h'Q$, where h is the height of the segment and $h' = A - h$. Then $P \ll S \ll Q$, whence (10) follows.

Using (10), we find from (9)

$$z_\Pi^0 \ll \frac{(n+1)\varkappa_{n+1}}{2\varkappa_n} B \left(\frac{A}{B}\right)^{n^2+n+1}. \tag{12}$$

By the method which we used in the proof of Theorem 3 ([3]), one can show that

$$x(0) \gg A + \frac{\varkappa_{n+1}(n+1)}{\varkappa_n A^n}\left(B^{n+1} - A^{n+1}\right). \tag{13}$$

From inequalities (6), (8), (12), (13) the estimate (2) follows.

Proof of Proposition A. Denote by Π the center of gravity of the body bounded by Γ. Let $m = \min_L \rho(\Pi, L)$, and let L_m be a supporting plane for which $\rho(\Pi, L_m) = m$. Suppose that the x-axis is the outer normal to L_m. With the aid of Steiner symmetrizations and a limiting passage one can obtain from Γ a surface of revolution Σ about the x-axis. For the surface Σ the conditions of Lemma 3 are fulfilled. Indeed, $V_\Sigma = V_\Gamma$, and by Theorem 3 ([2]) $V_\Gamma \gg \varkappa_{n+1}B^{n+1}$. Moreover, by Theorem 4 ([2]) the Gaussian curvature of Σ is not less than a^n. Denote by $Z = (z, 0, \dots, 0)$ the center of gravity of the body bounded by Σ, and by L_Σ the supporting plane to Σ for which the x-axis is the outer normal. Then $\rho(Z, L_\Sigma) = x(0) - z$. On the other hand, one can show that $\rho(Z, L_\Sigma) = \rho(\Pi, L_m)$. Consequently, $m = x(0) - z$. But $r \gg m_1$, since otherwise a ball of radius $\bar{r} > r$ could be inscribed in T ([1], p. 24). Thus,

$$r \gg x(0) - z.$$

4. The assertion of Theorem 1 concerning the radius R follows from the following estimate.

Proposition B. *If the Gaussian curvature K of the convex surface Γ satisfies the inequalities $a^n \ll K \ll b^n$, then for the radius R of the minimal circumscribed ball the estimate*

$$R \ll 2B - A + \frac{B\varkappa_{n+1}}{\varkappa_n}\left[\left(\frac{A}{B}\right)^{(n+1)^2} - 1\right] +$$
$$+ \left(A^{n+1} - B^{n+1}\right)\frac{(\chi+1)\varkappa_{n+1}}{\varkappa_n}\left[\frac{1}{A^n} + \frac{1}{B^n}\left(\frac{A}{B}\right)^{n^2+n+1}\right].$$

Proof. If $\bar{R}$ is the radius of the minimal circumscribed ball whose center coincides with the center of the maximal inscribed ball, then $D \gg r + \bar{R}$ and $R \ll \bar{R}$.

Using estimate (1) for r and the estimate for D in Theorem 2 ([3]), we obtain the estimate in Proposition B.

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Note: Figure translations are in progress. See original paper for figures.