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Abstract

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PROJECTIONS OF SEMILATTICES

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A **projection** (projective mapping) of a semilattice* Γ onto a semilattice Γ' is an isomorphism of the structure of all subsemilattices of Γ (the empty set is also regarded as a subsemilattice) onto the structure of all subsemilattices of Γ'. The concept of projection applies to an arbitrary class of algebraic systems. At the same time the term "structural isomorphism" is no less widespread. It is clear that in the case of semilattices, and especially of lattices, the term "projection" is preferable.

In the literature, projections of groups (^{1}, ^{2}), semigroups (^{3}), linear varieties over a field or, more generally, modules over regular rings (^{4}, ^{5}) (in the cited works see the corresponding extensive bibliography), and certain other algebraic systems have been studied. The basic questions in the theory of projections are the following, which for definiteness we formulate for semilattices.

a) Finding necessary and sufficient conditions that must be satisfied by two arbitrary semilattices which are projected onto one another; in other words, finding all projections of an arbitrary semilattice.

b) Describing all semilattices that are projected only onto isomorphic semilattices.

Every isomorphism of a semilattice induces its projection. The converse (as for many other algebraic systems), generally speaking, is false. Therefore there also arises the following important question, adjacent to question b):

c) Describing all semilattices every projection of which is induced by some isomorphism.

Solutions (complete and partial) of the questions formulated are known for various classes of algebraic systems. Thus, for example, question a) has been completely solved for finite groups (^{1}). Many sufficient criteria have been established for systems satisfying the conditions of questions b) and c). We mention, in particular, the first fundamental theorem of projective geometry (^{4}) (and its generalization in (^{6})). Various classes of groups (semigroups) determined by the structure of their subgroups (subsemigroups), etc., have been found (for details on the corresponding results see the works cited above). In the case of modules over a ring, semilinear mappings play the role of isomorphisms.

In the present note a solution of questions a)–c) is given for semilattices. Since, allowing some liberty, one may say that the class of semilattices coincides with the class of commutative semigroups of idempotents (for the precise meaning of the relations between these classes see ([7]), Ch. II), the same questions are thereby automatically solved for the latter, and one can even consider—

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[*] Everywhere below, by a semilattice we mean a semilattice with respect to intersection, i.e. a partially ordered set in which every pair of elements has a greatest lower bound.

consider projections of commutative semigroups of idempotents onto commutative semigroups, for a semigroup onto which a semigroup of idempotents is projected will itself, as is easy to see, be a semigroup of idempotents.

Let us give some definitions. Everywhere below the letter Γ denotes an arbitrary fixed semistructure. By the letter σ we shall denote the binary comparability relation in Γ, i.e., the union of the relations $\leqslant$ and $\geqslant$; by $\bar{\sigma}$ we denote the complementary incomparability relation.

A subsemistructure $H \subseteq \Gamma$ will be called **homogeneous** if either $H = \Gamma$, or, for an arbitrary $x \in \Gamma \setminus H$, the following alternative holds:

either

$$x\sigma h \quad \text{for every } h \in H, \tag{1}$$

or

$$x\bar{\sigma}h \quad \text{for every } h \in H. \tag{2}$$

A subsemistructure $H \subseteq \Gamma$ will be called **completely isolated** (as in (7) for subsemigroups) if $\Gamma \setminus H$ is a subsemistructure. If H is a homogeneous completely isolated (h.c.i.) subsemistructure of Γ, then by $\Gamma_\sigma(H)$ we shall denote the set of all elements of $\Gamma \setminus H$ satisfying condition (1), and by $\Gamma_{\bar{\sigma}}(H)$ the set of all elements of $\Gamma \setminus H$ satisfying condition (2). We have

$$\Gamma_\sigma(H) \cup \Gamma_{\bar{\sigma}}(H) = \Gamma \setminus H, \qquad \Gamma_\sigma(H) \cap \Gamma_{\bar{\sigma}}(H) = \emptyset.$$

An h.c.i. subsemistructure $H \subseteq \Gamma$ will be called **saturated** if for every h.c.i. subsemistructure $F \subseteq \Gamma$ strictly containing H, the strict inclusion $\Gamma_\sigma(F) \subseteq \Gamma_\sigma(H)$ holds. A one-to-one mapping φ of the semistructure Γ onto the semistructure Γ' will be called a **weak isomorphism** if, for arbitrary $x, y \in \Gamma$, the following conditions are fulfilled: 1) $x\sigma y \leftrightarrow \varphi(x)\sigma\varphi(y)$ (i.e., φ is an isomorphism of Γ as a σ-structure in the sense of Bourbaki (9)), and 2) $x\bar{\sigma}y \to \varphi(x \wedge y) = \varphi(x) \wedge \varphi(y)$.

It is obvious that the inverse mapping φ^{-1} will be a weak isomorphism of Γ' onto Γ. The notion of an ordinal sum of partially ordered sets is regarded as known (see (8)). Below, speaking of the decomposition of a semistructure into an ordinal sum, we shall always mean an ordinal sum of subsemistructures; and a semistructure possessing a decomposition into an ordinal sum of its own subsemistructures will be called **decomposable**, and otherwise **indecomposable**. An arbitrary semistructure has a unique **irreducible decomposition into**

an ordinal sum, i.e., such a decomposition whose components are no longer decomposable.

A preliminary answer to question a) is given by the following

Proposition 1. *The semistructures* Γ *and* Γ' *are projectable onto one another if and only if they are weakly isomorphic; moreover every projection of* Γ *onto* Γ' *is induced by some weak isomorphism of* Γ *onto* Γ'*.*

Thus it is necessary to clarify the mechanism of action of a weak isomorphism. To formulate the corresponding result we shall need several auxiliary propositions.

Lemma 1. *Let* H *be a homogeneous completely isolated subsemistructure of* Γ*. For every semistructure* T *there exists a semistructure* Γ' *such that* Γ' *contains a homogeneous completely isolated subsemistructure* H'*, isomorphic to* T*, and moreover*

$$\Gamma \setminus H = \Gamma' \setminus H'^{*}, \qquad \Gamma_{\bar\sigma}(H) = \Gamma'_{\bar\sigma}(H),$$

and for arbitrary $x \in \Gamma \setminus H$*, from the fact that* x *is less (greater) in* Γ *than any element of* H*, it follows that* x *is less (greater) in* Γ' *than any element of* H'*. If, moreover,* T *is weakly isomorphic to* H*, then every semistructure* Γ' *with the indicated property is weakly isomorphic to* Γ*.*

Lemma 2. *If, under the hypotheses of the preceding lemma,* H *is, in addition, a saturated subsemistructure, then, up to isomorphism, there exists only one semistructure* Γ' *with the property indicated in the lemma.*

* Equality of semistructures, not merely equality of sets.

Let T be a semistructure obtained from the semistructure H by some permutation of the components of some decomposition of H into an ordinal sum. It can be shown that H is projected onto T, and therefore H and T are weakly isomorphic by virtue of Proposition 1. If here H and T are semistructures from the hypotheses of Lemma 1, then we shall say that the semistructure Γ' is obtained from Γ by a permutation of components of some decomposition of H into an ordinal sum; in the case when H is saturated, such a semistructure, according to Lemma 2, is determined by the given permutation uniquely up to isomorphism. By what has been said, Lemma 1 implies the following.

Lemma 3. *A semistructure obtained from* Γ *by permuting the components of some decomposition into an ordinal sum of an arbitrary homogeneous completely isolated subsemistructure of it is weakly isomorphic to* Γ*.*

It is clear that, without loss of generality, one may consider only permutations of the components of an irreducible decomposition of a h.c.i. subsemistructure into an ordinal sum. This is useful to keep in mind in what follows.

The assertion of Lemma 3 can be generalized to the case of an arbitrary system of saturated h.c.i. subsemistructures of Γ. We note that the saturation requirement just mentioned is essential in this case. Namely, a component of an irreducible decomposition into an ordinal sum of an unsaturated h.c.i. subsemistructure H may turn out to be a component of an analogous decomposition of a h.c.i. subsemistructure F different from H; therefore not all permutations of the components of H and F will be possible simultaneously. For saturated h.c.i. subsemistructures, however, the situation is different, as follows from the next lemma.

Lemma 4. *If H and F are arbitrary distinct saturated homogeneous completely isolated subsemistructures of Γ, then either H and F do not intersect, or one of them is strictly contained in some component of an irreducible decomposition of the other into an ordinal sum.*

For the generalization of Lemma 3 indicated above, it is necessary to justify the correctness (to give an exact definition) of the notion of a semistructure obtained from a given semistructure by arbitrary permutations of the components of irreducible decompositions into an ordinal sum of certain saturated h.c.i. subsemistructures (of course, only the case of an infinite system of such subsemistructures needs justification; in the finite case Lemma 3 is simply applied several times). This can be done as follows. For a given system $[H_\alpha]$ of saturated h.c.i. subsemistructures of Γ (α ranges over some index set A), let $[\psi_\alpha]$ denote the corresponding system of permutations: for any $\alpha \in A$, ψ_α is a one-to-one mapping of the ordered set of components of the irreducible decomposition of H_α into an ordinal sum onto the set of the same components, arranged in a new order. We shall say that elements $x, y \in \Gamma$ participate in the permutation ψ_α if they belong to distinct components of the indicated decomposition of H_α. On the set Γ introduce the following binary relation μ.

If x and y do not participate in any permutation from $[\psi_\alpha]$, then set

$$x\mu y \quad \text{if and only if} \quad x \leqslant y.$$

If x and y participate in the permutation ψ_α, then set

$$x\mu y \quad \text{if and only if, as a result of the}$$

$$\text{permutation } \psi_\alpha, \text{ the component containing } x$$

$$\text{becomes a component preceding the component containing } y.$$

Since, by Lemma 4, arbitrary elements $x, y \in \Gamma$ can participate in no more than one permutation, the relation μ is well defined. It can be verified that μ is a partial-order relation and, moreover, that Γ with respect to μ is a semistructure. Denote this semistructure by

Here, by Γ', and we shall say that Γ' is **obtained from the semistructure** Γ **by the permutations** $[\psi_\alpha]$. Γ' is uniquely determined by the given permutations.

Proposition 2. *If the semistructure* Γ' *is obtained from* Γ *by the permutations* $[\psi_\alpha]$*, then the mapping that assigns to any* $x \in \Gamma$ *the same element, regarded as an element of* Γ'*, is a weak isomorphism from* Γ *onto* Γ'*.*

The weak isomorphism referred to in Proposition 2 will be called **canonical**. An important role in the solution of question a) is played by

Proposition 3. *Up to isomorphism, every weak isomorphism is canonical.*

From Propositions 1–3 there follows the following theorem, which completely solves question a).

Theorem A. *In order that a semistructure* Γ *be projectable onto a semistructure* Γ'*, it is necessary and sufficient that* Γ' *be isomorphic to a semistructure obtained from* Γ *by certain permutations of the components of the irreducible decomposition into an ordinal sum of some of its saturated homogeneous completely isolated subsemistructures.*

To solve question b), it is therefore necessary to describe all semistructures which pass, as a result of the indicated permutations, into semistructures isomorphic to themselves. It can be shown that any such semistructure has the following properties: 1) the irreducible decomposition into an ordinal sum of each of its homogeneous completely isolated subsemistructures consists of a finite number of isomorphic components; 2) it satisfies the condition of termination of increasing chains decomposable into an ordinal sum of homogeneous completely isolated subsemistructures.

The necessary and sufficient criterion solving question b) consists in the conjunction of these two conditions and one further condition, the formulation of which would require some additional definitions (so that we omit here the exact formulation of the corresponding **Theorem B**). The criterion mentioned, being rather complicated in the general case, is considerably simplified for many classes of semistructures. We shall give only one example of this kind.

Theorem B′. *In order that a semistructure all of whose chains are finite (in particular, a finite semistructure) have projections only onto isomorphic semistructures, it is necessary and sufficient that the irreducible decomposition into an ordinal sum of any of its homogeneous completely isolated subsemistructures consist of isomorphic components.*

The following theorem completely solves question c).

Theorem C. *In order that every projection of a given semistructure be induced by some isomorphism of it, it is necessary and sufficient that all homogeneous completely isolated subsemistructures of this semistructure be indecomposable into an ordinal sum.*

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