A. A. GOLDBERG, B. Ya. LEVIN

1964

SovietRxiv

Abstract

Full Text

A. A. GOLDBERG, B. Ya. LEVIN

ON ENTIRE FUNCTIONS BOUNDED ON THE REAL AXIS

(Presented by Academician S. N. Bernstein on 8 January 1964)

It is known ([1], p. 71) that an entire function $f(z)$ $(z = x + iy = re^{i\varphi})$ of finite degree σ, $0 < \sigma < \infty$, satisfying on the real axis the inequality $|f(x)| \leq A$, necessarily satisfies the inequality $|f(z)| \leq A \exp(\sigma|y|)$, $|z| < \infty$. For the function $f(z) = A \exp(-i\sigma z)$ for $y \geq 0$ we have exact equality in the preceding inequalities. In the present note we consider the estimate of an entire function $f(z)$ of finite degree σ, if the condition $|f(x)| \leq A$ is replaced by the conditions $|f(x)| \leq A$ for $x < 0$ and $|f(x)| \leq B$ for $x > 0$. This problem is included in the list of current problems compiled by the participants of the colloquium on the classical theory of functions held in August 1961 at Cornell University (USA) ([2], [3]). We shall prove the following theorem.

Theorem. Let the entire function $f(z)$ of finite degree σ, $0 < \sigma < \infty$, satisfy the conditions: $|f(x)| \leq A$ for $x < 0$ and $|f(x)| \leq B$ for $x > 0$. Then, for $|z| < \infty$, the inequality holds $(0 \leq |\varphi| \leq \pi)$

$$\left|f\left(re^{i\varphi}\right)\right| \leq B(A/B)^{|\varphi|/\pi} e^{\sigma|y|} = \Omega\left(re^{i\varphi}, A, B\right) e^{\sigma|y|}. \tag{1}$$

On the other hand, one can exhibit an entire function $f(z)$ satisfying the conditions of the theorem such that, in the upper half-plane, as $r \to \infty$, the ratio

$$\left|f\left(re^{i\varphi}\right)\right| / \Omega\left(re^{i\varphi}; A, B\right) e^{\sigma|y|} \to 1 \tag{2}$$

uniformly with respect to φ, $0 \leq \varphi \leq \pi$.

Proof. In the case $A = B$, as noted, inequality (1) is known, and the second assertion of the theorem can be replaced by a stronger one. Let now $A \neq B$. Without loss of generality, we may assume that $B > A > 0$. In the entire finite z-plane the estimate $|f(z)| \leq B \exp(\sigma|y|)$ is valid. Therefore the function, analytic in the upper half-plane,

$$\omega(z) = B^{-1} f(z) \exp\left\{i\left[\sigma z + \frac{\ln z}{\pi} \ln(A/B)\right]\right\}$$

is bounded for $y > 0$. It is easy to see that $|\omega(x)| \leq 1$, $-\infty < x < +\infty$, $x \neq 0$. By the Phragmén-Lindelöf principle it follows that $|\omega(z)| \leq 1$ for $y > 0$. Consequently, estimate (1) is valid for $y > 0$. For $y < 0$ analogous arguments are carried out.

We turn to the proof of the second assertion of the theorem. First of all, note that it suffices for us to construct an example of a function $f(z)$ of finite degree σ for which relation (2) would hold for one particular σ, $0 < \sigma < \infty$. Indeed, in order to obtain an analogous example with finite degree σ_1, $0 < \sigma_1 < \infty$, one may consider the function $f_1(z) = f(\sigma_1 z/\sigma)$. Choose a natural number n such that

$$n > 3(B - A)\ln 2\,\{2\pi A\}^{-1}.$$

Denote by $\Phi(z)$ the function

$$\Phi(z) = \int_{C_z} \left\{\frac{1}{nt^3} + \frac{1}{(t - \pi n)^4}\right\} \sin^4 t\, dt,$$

where C_z is the curve consisting of the interval of the real axis $(-\infty, x)$ and the segment joining the points x and $x + iy$. It is not difficult to verify that $\Phi(z)$ is an entire function of finite degree 4 with indicator $h(\varphi; \Phi) = 4|\sin\varphi|$ (the latter assertion follows from the boundedness of $|\Phi(x)|$ and the convexity of the indicator diagram; one may also use the known relations between the indicators of an entire function and its derivative). We note that for $-\infty < x < +\infty$ one has $(x - \pi n)^4 + nx^3 > 0$; this is easily checked by putting $x = nx$. Then it is easy to see that the function $\Phi(z)$, which assumes real values on the real axis, decreases from 0 to $\Phi(0)$,

$$0 > \Phi(0) > \int_{-\infty}^{0} \frac{\sin^4 t}{nt^3}\, dt = -\frac{\ln 2}{n},$$

as $z = x$ increases from $-\infty$ to 0, and, as x increases from 0 to $+\infty$, the function $\Phi(x)$ increases from $\Phi(0)$ to

$$\int_{-\infty}^{+\infty} \left\{\frac{1}{nt^3} + \frac{1}{(t - \pi n)^4}\right\} \sin^4 t\, dt = 2\int_0^{\infty} \frac{\sin^4 t}{t^4}\, dt = \frac{2}{3}\pi.$$

Let

$$\psi(z) = A + \frac{3(B - A)}{2\pi}\Phi(z).$$

Obviously, on the real axis $0 < \psi(x) < B$, and for $x < 0$ one has $\psi(x) < A$; moreover, the entire function $\psi(z)$ is of finite degree 4 with indicator $h(\varphi; \psi(z)) = 4|\sin\varphi|$. Let $\chi(z)$ be the Blaschke product for the half-plane, formed from the zeros of the function $\psi(z)$ lying in the upper half-plane. It converges, since the function $\psi(z)$ is a function of class A ((1), Ch. V). Then $f(z) = \psi(z)[\chi(z)]^{-1}$ is an entire function without zeros for $y \geqslant 0$, and $|f(x)| = \psi(x)$. Since the estimate $|\ln|\chi(re^{i\varphi})|| = o(r)$ holds for all $r > 0$, apart from a set of finite logarithmic length (4), it follows (see (1)) that $h(\varphi; f) = 4|\sin\varphi|$. The harmonic function

$\ln|f(z)|$ for $y \geqslant 0$ admits in the upper half-plane the integral representation (([1]), p. 311)

$$\ln|f(z)| = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{\ln|f(t)|\,dt}{(t-x)^2+y^2} + 4y.$$

Then ($y \geqslant 0$)

$$\ln|f(z)| - 4y - \ln\Omega(z;A,B) =$$

$$= \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{\ln|f(t)|\,dt}{(t-x)^2+y^2} - \frac{y}{\pi}\int_{-\infty}^{0}\frac{\ln A\,dt}{(t-x)^2+y^2} - \frac{y}{\pi}\int_{0}^{\infty}\frac{\ln B\,dt}{(t-x)^2+y^2} =$$

$$= -\frac{y}{\pi}\int_{-\infty}^{0}\frac{\ln(A/\psi(t))}{(t-x)^2+y^2}\,dt - \frac{y}{\pi}\int_{0}^{\infty}\frac{\ln(B/\psi(t))}{(t-x)^2+y^2}\,dt \geqslant$$

$$\geqslant -\frac{y}{\pi}\int_{-\infty}^{-\sqrt{r}}\frac{\ln(A/\psi(-\sqrt{r}))}{(t-x)^2+y^2}\,dt - \frac{y}{\pi}\int_{-\sqrt{r}}^{\sqrt{r}}\frac{\ln(B/\psi(0))}{(t-x)^2+y^2}\,dt - \frac{y}{\pi}\int_{\sqrt{r}}^{\infty}\frac{\ln(B/\psi(\sqrt{r}))}{(t-x)^2+y^2}\,dt \geqslant$$

$$\geqslant -\ln\left\{\max\left(\frac{A}{\psi(-\sqrt{r})},\frac{B}{\psi(\sqrt{r})}\right)\right\}\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{dt}{(t-x)^2+y^2} - \ln\left(\frac{B}{\psi(0)}\right)\frac{1}{\pi}\arg\frac{z-\sqrt{r}}{z+\sqrt{r}} \geqslant$$

$$\geqslant -\ln\left\{\max\left(\frac{A}{\psi(-\sqrt{r})},\frac{B}{\psi(\sqrt{r})}\right)\right\} - \ln\left(\frac{B}{\psi(0)}\right)\frac{2}{\pi}\operatorname{arc\,tg}\frac{1}{\sqrt{r}}.$$

Since, as $r \to +\infty$, we have $\psi(-\sqrt{r}) \to A$, $\psi(\sqrt{r}) \to B$, using inequality (1) (with $\sigma = 4$), we easily obtain relation (2).

Remark. Using Hayman' s result ([4]), it is not hard to show that (2) also holds in the lower half-plane if, in passing to the limit $r \to +\infty$, one discards a certain set of values of r of finite logarithmic length.

Lviv State University
named after I. Franko

Kharkov State University
named after A. M. Gorky

Received
20 I 1964

REFERENCES

[1] B. Ya. Levin, *Distribution of zeros of entire functions*, Moscow, 1956.
[2] Bull. Am. Math. Soc., **68**, No. 1, 21 (1962).
[3] Sborn. per. Matematika, **7**, No. 5, 133 (1963).
[4] W. K. Hayman, J. math. pures et appl., **35**, 115 (1956).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.