M. I. ZHALDAK

1.** The algorithm presented here for solving the problem of finding

1964

Abstract

Full Text

M. I. ZHALDAK

ON THE CHEBYSHEV APPROXIMATION OF A CONTINUOUS FUNCTION BY A POLYNOMIAL IN THE PRESENCE OF CONSTRAINTS ON THE COEFFICIENTS

(Presented by Academician A. Yu. Ishlinskii, June 1, 1964)

1. The algorithm presented here for solving the problem of finding

$$\min_{\xi\in\Omega}\ \max_{t\in Q}\left|\sum_{k=1}^{n}\xi_k\varphi_k(t)-f(t)\right|, \tag{1}$$

where Ω is either the entire n-dimensional space or a convex set defined by the constraints

$$\eta_i(q)=\sum_{k=1}^{n}\xi_k\varphi_{ki}(q)+b_i(q)\geq 0 \quad (q\in Q_i;\ i=1,\dots,p); \tag{2}$$

$\varphi_k(t),\ \varphi_{ki}(q)$ $(k=1,\dots,n;\ i=1,\dots,p)$ are a given system of real continuous functions; Q, Q_i $(i=1,\dots,p)$ are given compact sets ([1,2]), is constructed on the ideas of the algorithms ([2,3]).

Problem (1) arises, for example, in solving the problem (similar to the problems considered in ([5,6])) of the approximate construction of optimal controls for a process described by the equation

$$\frac{\partial f(x,t)}{\partial t}=\frac{\partial^2 f(x,t)}{\partial x^2}+u_1(t) \quad (t>0;\ 0<x<l),$$

with initial and boundary conditions: $f(x,0)=f_1(x)$, $f(0,t)=f_2(t)$, $f(l,t)=u_2(t)$. Here $u_1(t)$, $u_2(t)$ are controls subject to the constraints $|u_1(t)|\leq M_1$, $|u_2(t)|\leq M_2$; $f_1(x)$, $f_2(t)$, and the function $f_3(x)$ introduced below are given fixed functions. The optimality of the controls consists in the fact that at the moment $t=T$ the quantity

$$\min_{u_1,u_2}\ \max_{0\leq x\leq l}|f(x,T)-f_3(x)|$$

must be attained.

If the controls $u_1(t)$, $u_2(t)$ are replaced by the previously given functions $g_i(t)$, $h_k(t)$ $(i = 1, \dots, m;\ k = 1, \dots, n-m)$, and if the solutions $f(x,t)\big|_{t=T}$ of the differential equation for $f_1(x) \equiv 0$, $f_2(t) \equiv 0$, $u_1(t) = g_i(t)$, $u_2(t) \equiv 0$ are denoted by $S_i(x)$, and respectively for $f_1(x) \equiv 0$, $f_2(t) \equiv 0$, $u_1(t) \equiv 0$, $u_2(t) = h_k(t)$ by $P_k(x)$, then to the polynomials

$$\tilde{u}_1(t) = \xi_1 g_1(t) + \dots + \xi_m g_m(t), \qquad \tilde{u}_2(t) = \xi_{m+1} h_1(t) + \dots + \xi_n h_{n-m}(t),$$

approximating the controls, there will correspond the polynomial

$$\xi_1 S_1(x) + \dots + \xi_m S_m(x) + \xi_{m+1} P_1(x) + \dots + \xi_n P_{n-m}(x),$$

whose coefficients must be chosen so that the following is attained:

$$\min_{\xi\in\Omega} \max_{0\le x\le l} \left| \sum_{k=1}^{m} \xi_k S_k(x) + \sum_{k=m+1}^{n} \xi_k P_{k-m}(x) - F(x) \right|,$$

where Ω is determined by the conditions

$$\left| \sum_{k=1}^{m} \xi_k g_k(t) \right| \le M_1, \qquad \left| \sum_{k=m+1}^{n} \xi_k h_{k-m}(t) \right| \le M_2,$$

and $F(x)$ is a known function.

2. By introducing an additional variable, problem (1)–(2) is reduced to a linear programming problem of the following form: minimize the linear form $z = p_1\xi_1 + \dots + p_n\xi_n$ subject to the conditions

$$\sum_{k=1}^{n} \psi_{kj}(q)\xi_k + M_j(q) \ge 0 \qquad (q \in Q_j;\ j = 1, \dots, m). \tag{3}$$

The latter problem is solved by an algorithm using the computational scheme of the simplex method (see, for example, (8)) and based on one special method which, using the continuity of the specification of the constraints, makes it possible to construct a discrete ε-net only for those parts of the compact sets Q_j whose consideration is sufficient for the approximate construction of the solution of the problem (starting from an arbitrary point), without affecting the remaining parts of these compact sets. Thus the "unequal rights" of the points of the uniform ε-net of the compact sets Q_j are taken into account.

Let $m = 1$. Write the constraints (3) and the minimized linear form in the form of a table (up to the double dotted line)

	ξ_1	$\cdots$	ξ_n	1
$\eta(q)$	$\psi_1(q)$	$\cdots$	$\psi_n(q)$	$M(q)$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
z_1	p_1	$\cdots$	p_n	0
η_1	$\psi_1(q_1)$	$\cdots$	$\psi_n(q_1)$	$M(q_1)$

(4)

where $\eta(q)$ denotes the deviation of an arbitrary point ξ from the plane

$$\eta(q) \equiv \psi_1(q)\xi_1 + \cdots + \psi_n(q)\xi_n + M(q) = 0 \quad (q \in Q).$$

Using Jordan eliminations ((4), cf. also (3)), we first eliminate the variables $\xi_1, \ldots, \xi_n$. To eliminate one of them, take an arbitrary point $q_1 \in Q$, add to table (4) the row corresponding to this point, for which we compute $\psi_k(q_1)$ $(k = 1, \ldots, n)$, $M(q_1)$, and, denoting $\eta(q_1)$ by η_1, perform a step of Jordan elimination, interchanging η_1 with one of the ξ_k. Removing from the compact set Q the interior of some sphere of radius ρ with center at the point q_1 (the ρ-neighborhood of the point q_1), except for this point itself ([2]), take on the remaining compact set $Q^{(1)}$ an arbitrary point $q_2 \neq q_1$. To the table obtained from (4) after the Jordan elimination step, we add a new row corresponding to q_2, and move η_2 upward in the table, interchanging it with one of the remaining ξ_k. We proceed in this way until either all the ξ_k have been eliminated, or, opposite the remaining ξ_k, there are identically zero entries, so that in the remaining columns it will be impossible to find even a single nonzero resolving element ([3,4]).

Without loss of generality, we shall assume that it has been possible to eliminate all ξ_k, so that after n steps of Jordan eliminations there is obtained, for example, the table (up to the double dotted line):

	η_1	$\cdots$	η_s	$\cdots$	η_n	1
$\eta(q)$	$\psi_1^{(n)}(q)$	$\cdots$	$\psi_s^{(n)}(q)$	$\cdots$	$\psi_n^{(n)}(q)$	$M^{(n)}(q)$
$\cdots$	$\cdots$		$\cdots$		$\cdots$	$\cdots$
z	$p_1^{(n)}$	$\cdots$	$p_s^{(n)}$	$\cdots$	$p_n^{(n)}$	$z^{(n)}$
$\cdots$	$\cdots$		$\cdots$		$\cdots$	$\cdots$
ξ_k	$\tilde{\psi}_1^{(n)}(q_k)$	$\cdots$	$\tilde{\psi}_s^{(n)}(q_k)$	$\cdots$	$\tilde{\psi}_n^{(n)}(q_k)$	$\tilde{M}^{(n)}(q_k)$
$\cdots$	$\cdots$		$\cdots$		$\cdots$	$\cdots$
η_j	$\psi_1^{(n)}(q_j)$	$\cdots$	$\psi_s^{(n)}(q_j)$	$\cdots$	$\psi_n^{(n)}(q_j)$	$M^{(n)}(q_j)$

(5)

If in table (5) $M^{(n)}(q) \geq 0$ for all $q \in Q^{(n)}$, where $Q^{(n)}$ is obtained from Q after discarding the ρ-neighborhoods of the points $q_1, \ldots, q_n$, with the exception of these points themselves, then we have a supporting point $\eta_1 = 0, \dots, \eta_n = 0$ of the domain Ω, determined by the conditions (3) for $q \in Q^{(n)}$.

Suppose that for some $q \in Q^{(n)}$ in (5) $M^{(n)}(q) < 0$. Then take one of the functions $\psi_k^{(n)}(q)$ $(k = 1, \dots, n)$ that assumes positive values for some of these q, for example the function $\psi_s^{(n)}(q)$ (if no such function exists, then the constraints are incompatible), and find on $Q^{(n)}$ (excluding the points corresponding to the deviations standing at the top of the table) the smallest, in absolute value, nonnegative value of the ratio

$$\frac{M^{(n)}(q)}{\psi_s^{(n)}(q)}. \tag{6}$$

Let it be attained at the points $q_{n+1}, \dots, q_{n+l}$.* Then we append to table (5) the rows corresponding to the new points; for this purpose we compute the values of the functions $\psi_k^{(n)}(q)$, $M^{(n)}(q)$ at the points $q_{n+1}, \dots, q_{n+l}$, choose a pivot element in one of these rows in the column marked earlier, and perform a Jordan elimination step. We proceed in this way until a supporting point of the domain Ω is found or the incompatibility of the constraints (3) is discovered. Degeneracy (the ratio (6) is equal to zero) is overcome by one of the known techniques.

Suppose now that in table (5) (up to the double dotted line) $M^{(n)}(q) \geq 0$ for all $q \in Q^{(n)}$. If, moreover, $p_k^{(n)} \geq 0$ $(k = 1, \dots, n)$, then the point $\eta_1 = 0, \dots, \eta_n = 0$ is optimal. If, for example, $p_s^{(n)} < 0$, then we find on $Q^{(n)}$ the points at which $\psi_s^{(n)}(q) < 0$ (if there are no such points, then the linear form is unbounded below), after which we find on $Q^{(n)}$ (excluding the points corresponding to the deviations standing at the top of the table) the smallest, in absolute value, nonnegative value of the ratio (6). If it is attained at the points $q_{n+1}, \dots, q_{n+l}$, then we append to table (5) the rows corresponding to these points, as before, then choose a pivot element in one of these rows and perform a Jordan elimination step. We proceed in this way until we either discover that the linear form is unbounded below (an optimal point does not exist), or find an optimal point.

In the case $m > 1$, the problem is solved analogously.

3. Problem (1) can also be solved directly, without passing to a linear programming problem. The algorithm for solving this problem in the case where Ω is the whole n-dimensional space is described in (7). In the presence of the constraints (2), in the tables considered in (7), in addition to the deviations $\delta(t) = \sum_{k=1}^{n} \varphi_k(t)\xi_k - f(t)$, the deviations (2) are also entered. The process begins with an arbitrary point $\xi^{(1)} \in \Omega$. If at this point, for some $q \in Q_i$, the deviations $\eta_i(q)$ are equal to zero, then the corresponding rows are appended to the table. By means of Jordan elimination steps we move to the top of the table all maximal deviations δ and all annulling deviations η. Suppose that at the top of the table there are $\delta_1, \dots, \delta_p, \eta_1, \dots, \eta_l, \xi_{p+l+1}, \dots, \xi_n$. From the point $\xi^{(1)}$ we move along the line $\delta_1 = \ldots = \delta_p, \eta_1 = 0, \ldots, \eta_l = 0, \xi_{p+l+1} = \xi^{(1)}_{p+l+1}, \ldots, \xi_n = \xi^{(1)}_n$. The step size $\xi^{(r)}_{\min}$ in this direction is determined analogously to how this is done in (7), and in such a way as not to leave the domain Ω.

* If this smallest ratio is attained on an infinite set of points of the compact set $Q^{(n)}$, then as $q_{n+1}, \ldots, q_{n+l}$ we take the centers of spheres forming a finite ρ-covering of this set.

By slightly modifying the definition of the characteristic of an edge, it is easy to obtain an optimality criterion for a stationary point ([3],[7]).

Remark 1. In exactly the same way, the finite algorithm ([8]) is applied to the approximate solution of the quadratic programming problem, i.e., the problem of minimizing the quadratic function

$$f(x) \equiv \sum_{i,k=1}^{n} a_{ik} x_i x_k + \sum_{i=1}^{n} b_i x_i + c,$$

where

$$\sum_{i,k=1}^{n} a_{ik} x_i x_k$$

is a positive definite quadratic form, under continuously specified constraints

$$\sum_{i=1}^{n} \varphi_i(t) x_i + b(t) \geq 0 \quad (t \in Q).$$

Let us note that this problem can be reduced to the linear programming problem considered in Sec. 2.

Remark 2. The algorithm presented is applicable to the problem of finding the Chebyshev point of the system of constraints (2), i.e., the point ξ^* for which

$$L = \inf_{\xi} \max_{i,\ q \in Q} \eta_i(q, \xi)$$

is attained ([3]).

I consider it my pleasant duty to express my deep gratitude to Prof. S. I. Zukhovitskii and G. I. Eskin for their attention to this work.

Kyiv State Pedagogical Institute
named after A. M. Gorky

Received
18 IV 1964

CITED LITERATURE

[1] E. G. Golshtein, *DAN*, **140**, No. 1, 23 (1961).
[2] S. I. Zukhovitskii, *DAN*, **120**, No. 4, 693 (1958).
[3] S. I. Zukhovitskii, *DAN*, **139**, No. 3, 534 (1961).
[4] E. Stiefel, *Numer. Math.*, **2**, 1 (1960).
[5] Yu. V. Egorov, *DAN*, **145**, No. 4, 720 (1962).
[6] A. G. Butkovskii, A. Ya. Lerner, *Avtomatika i telemekh.*, **21**, 6, 682 (1960).
[7] M. I. Zhaldak, *Tr. Kievsk. vyssh. inzh. radiotekhnich. uchilishcha*, **32**, 32 (1962).
[8] S. I. Zukhovitskii, L. I. Avdeeva, *Linear and Convex Programming*, "Nauka," 1964.

Note: Figure translations are in progress. See original paper for figures.