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Abstract

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A Non-improvable Estimate of Trigonometric Sums for Recurrent Functions with Nonconstant Coefficients

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Notation. K is a field of algebraic numbers of finite degree over the field of rational numbers; ρ is a prime ideal of the field K; $N(\rho)$ is its norm; P is the residue class field of the integers of the field K modulo ρ; $\mathfrak{d}$ is the different of the field K; λ is an ideal relatively prime to the ideal $\mathfrak{d}\rho$ and belonging to the same class to which the ideal $\mathfrak{d}\rho$ belongs; $\omega = \lambda/\mathfrak{d}\rho$; $S(\alpha)$ is the trace of the number α from the field K; n and q are natural numbers; $a_1(x), \dots, a_n(x)$ are functions periodic modulo ρ, taking, for all natural values of the argument, values that are integers of the field K; q is the primitive period modulo ρ of the vector-function $(a_1(x), \dots, a_n(x))$, i.e. the least natural number with the condition that, for all natural x, the congruences

$$a_1(x+q) \equiv a_1(x) \pmod{\rho}, \dots,\ a_n(x+q) \equiv a_n(x) \pmod{\rho}$$

hold.

Let $a_1(x) \not\equiv 0 \pmod{\rho}$ for all natural x; consider the recurrence equation over the field K modulo ρ, of order n,

$$\psi(x) = a_1(x)\psi(x-n) + \dots + a_n(x)\psi(x-1) \tag{1}$$

with coefficients $a_1(x), \dots, a_n(x)$. If $\psi(1), \dots, \psi(n)$ are arbitrary integers of the field K, then equation (1) determines a single-valued function $\psi(x)$ in the field K, defined for all natural x. We shall call this function a **solution of equation** (1). We shall assume that the numbers $\psi(1), \dots, \psi(n)$ are not all multiples of ρ. Denote by δ_x, for each natural x, the residue class modulo ρ of the integers of the field K with representative $\psi(x)$. The sequence

$$\delta_1, \delta_2, \dots, \delta_x, \dots \tag{2}$$

will be called the recurrent modulo ρ sequence corresponding to the solution $\psi(x)$ of equation (1).

We shall say that two solutions of equation (1) are **equivalent** if, after discarding a certain number of the first terms of the recurrent sequence corresponding to one of these solutions, we obtain the recurrent sequence corresponding to the other solution. It is easy to see that there exists a finite set of pairwise inequivalent solutions of equation (1) with the condition that every solution of equation (1) is equivalent to one of the solutions of this set. Solutions of equation (1) taken from any one such set will be called **basic**.

In our first note on recurrent functions with nonconstant coefficients ([1]) it was established that the sequence (2) is periodic, that its period does not exceed $q(N(\rho)^n - 1)$, and the problem was considered of the existence of a recurrent sequence whose primitive period attains this value.

In the present note we consider the problem of an unimprovable estimate for the modulus of the trigonometric sum $\sigma_y(\psi(x))$ of the form

$$\sigma_y(\psi(x)) = \sum_{x=1}^{y} \exp(2\pi i S(\psi(x)\cdot\omega)).$$

For recurrent functions over the field of rational numbers and with constant coefficients such a problem was solved by N. M. Korobov ([2]), and for recurrent functions over the field of algebraic numbers with constant coefficients—by L. L. Stepanova.

We have obtained the following results:

Theorem 1. *Let all roots of the characteristic polynomial of equation (1) (the meaning of the term is established in note ([1])) be distinct and of one order t, $t > N(\mathfrak{p})^{n-1}$, and let $\psi(x)$ be a solution of equation (1) with greatest primitive period τ; then*

$$\left|\sigma_y(\psi(x))\right| \le \begin{cases} qN(\mathfrak{p})^{n/2}, & \text{if } y = \tau; \\ qN(\mathfrak{p})^{n/2}(1 + \ln q + n\ln N(\mathfrak{p})), & \text{if } 1 \le y < \tau. \end{cases} \tag{3}$$

Theorem 2. *Let $\varphi(x)$ be a polynomial of degree n over the field P with the condition that all its roots are distinct and of one order t, $t > qN(\mathfrak{p})^{n-1}$. If $q = 1$, or if q exceeds any of its natural divisors by no less than n, then there exists a recurrent equation over the field K modulo $\mathfrak{p}$ of order n such that:*

a) *the primitive period modulo $\mathfrak{p}$ of the vector-function of its coefficients is equal to q;*

b) *all roots of its characteristic polynomial (not necessarily equal to $\varphi(x)$) are distinct and of one order t;*

c) *the primitive period τ modulo $\mathfrak{p}$ of any of its solutions is equal to tq;*

d) *for its fundamental solutions* $\psi^{(\nu)}(x)$*, whose number* s*, by virtue of c), is equal to* $\frac{1}{t}(N(\mathfrak{p})^n - 1)$*, the identity holds*

$$\sum_{\nu=1}^{s} \left|\sigma_\tau(\psi^{(\nu)}(x))\right|^2 = q^2((s-1)t+1). \tag{4}$$

It is easy to observe that from identity (4), for $s = 2$, it follows that for one of the fundamental solutions of equation (1) the inequality

$$\left|\sigma_\tau(\psi^{(\nu)}(x))\right| > \tfrac{1}{2} q N(\mathfrak{p})^{1/2}$$

is valid.

Let $\psi_\nu(x)$ $(1 \le \nu \le n)$ be the solution of equation (1) with initial conditions:

$$\psi_\nu(x) = \begin{cases} 1, & \text{if } x = \nu; \\ 0, & \text{if } x \ne \nu, \quad 1 \le x \le n, \end{cases}$$

and let

$$\delta_1^{(\nu)},\ \delta_2^{(\nu)}, \dots, \delta_x^{(\nu)}, \dots$$

be the recurrent sequence corresponding to this solution. Let β_x be the n-dimensional vector over the field P of the form

$$\beta_x = \begin{pmatrix} \delta_x^{(1)} \\ \vdots \\ \delta_x^{(n)} \end{pmatrix}.$$

For the proof of the first theorem we derive the following lemma.

Lemma. Let $\psi(x)$ be a solution of equation (1) with primitive period τ. Then, for any integer q, the inequality

$$\left|\sum_{x=1}^{\tau} \exp\left(S(\psi(x)\cdot\omega) + \frac{ax}{\tau}\right)\right| \le q^{1/2} N(\rho)^{n/2} \sqrt{\frac{M(\tau)}{\tau}},$$

holds, where $M(\tau)$ is the least number satisfying the condition: the number of pairs of equal vectors, for any natural x, in the sequence $\beta_{x+1}, \dots, \beta_{x+\tau}$ does not exceed $M(\tau)$.

Moreover, in the proof of both theorems an essential role is played by the fact established by us that, for any natural x, each succeeding vector of the sequence

$\beta_x, \beta_{x+q}, \beta_{x+2q}, \ldots$ is obtained from the preceding one by applying some one and the same linear operator independent of x.

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REFERENCES

1. V. I. Nechaev, DAN, **152**, No. 2, 270 (1963).

2. N. M. Korobov, DAN, **88**, No. 4, 603 (1953).

Note: Figure translations are in progress. See original paper for figures.