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Abstract

Full Text

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CONDITIONS FOR THE NONDECOMPOSABILITY OF DISSIPATIVE VOLTERRA OPERATORS WITH FINITE-DIMENSIONAL IMAGINARY COMPONENT

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Let A be a simple Volterra dissipative* operator acting in a separable Hilbert space $\mathfrak{H}$. If

$$\operatorname{sp} \frac{A - A^*}{2i} = l < \infty$$

and $W(z)$ is the characteristic matrix-function (c.m.f.) of the operator A, then ([2])

$$\|W(z)\| \leq e^{2l|z|}. \tag{1}$$

M. S. Brodskii showed ([2]) that if equality is attained in estimate (1), then the operator A is nondecomposable. In the present article some necessary conditions for nondecomposability are established.

1. Let the imaginary component

$$\frac{A - A^*}{2i}$$

of the operator A be n-dimensional ($2 \leq n < \infty$). The c.m.f. of the operator A satisfies the following conditions:

1) $W(z)$ is an entire matrix-function of the complex variable z;
2) $W(z) = I + izH + \ldots$, where H is a nonzero Hermitian nonnegative matrix-function;
3) $W^*(z)W(z) = I$ $(\operatorname{Im} z = 0)$;
4) $W^*(z)W(z) \geq I$ $(\operatorname{Im} z < 0)$.

A matrix-function $W_1(z)$ is called a divisor of the c.m.f. $W(z)$ if $W(z)$ can be represented in the form

$$W(z) = W_2(z)W_1(z),$$

where $W_k(z)$ $(k = 1, 2)$ satisfy conditions 1)–4). By virtue of ([3]), the operator A is nondecomposable if and only if its c.m.f. is ordered.

Lemma 1. *If the characteristic matrix-function $W(z)$ of the operator A has a scalar divisor of the form*

$$W_0(z) = e^{il_0 z} I \quad (l_0 > 0),$$

then the operator A is decomposable.

Proof. It suffices to show that the c.m.f. $W(z)$ is not ordered, i.e., that there exist such divisors of it which are not divisors of each other. Such divisors, as is easy to see, are the matrix-functions

$$W_k(z) = e^{il_0 z} P_k \quad (k = 1, 2),$$

where P_1 and P_2 are projection matrices of the same order as $W(z)$, with

$$P_1 P_2 = 0.$$

For entire matrix-functions, as also for scalar functions, one can introduce the concepts of order and type of growth.

Lemma 2. *The order of growth of any characteristic matrix-function of the operator A is equal to 1, and the type of growth σ satisfies the inequality*

$$\frac{2l}{n} \leq \sigma \leq 2l.$$

Proof. It is not hard to prove that all c.m.f.' s of the operator A have the same growth. Therefore, without loss of generality, one may

* The terminology introduced in ([1]) is used in the article.

consider that the order of the c.m.f. $W(z)$ is equal to n. Denoting by $\lambda_1^2 \leqslant \lambda_2^2 \leqslant \cdots \leqslant \lambda_n^2$ the eigenvalues of the matrix $W^*(z)W(z)$ and taking into account that $\det W(z) = e^{2ilz}(1)$, we obtain

$$\|W(z)\|^2 = \|W^*(z)W(z)\| =$$

$$= \lambda_n^2 \geqslant \sqrt[n]{\lambda_1^2 \lambda_2^2 \cdots \lambda_n^2} = \sqrt[n]{\det[W^*(z)W(z)]} = e^{-\frac{4l}{n}\operatorname{Im} z}.$$

Thus, for all z the inequality

$$\|W(z)\| \geqslant e^{-\frac{2l}{n}\operatorname{Im} z} \tag{2}$$

holds.

The assertion of the lemma follows from inequalities (1) and (2).

Theorem 1. *If the growth type σ of the c.m.f. $W(z)$ of the operator A satisfies the inequality*

$$\frac{2l}{n} \leqslant \sigma < \frac{2l}{n-1},$$

then the operator A is not unicellular.

Proof. We shall assume that the order of the c.m.f. $W(z)$ is equal to n. Since $\|W(z)\| = 1$ ($\operatorname{Im} z = 0$), by generalizing to matrix functions a well-known theorem from the theory of entire scalar functions ([4], Chapter I, Theorem 22), we obtain the inequality

$$\|W(z)\| \leqslant e^{-\sigma \operatorname{Im} z} \qquad (\operatorname{Im} z \leqslant 0). \tag{3}$$

Let $\lambda_1^2 \leqslant \lambda_2^2 \leqslant \cdots \leqslant \lambda_n^2$ be the eigenvalues of the matrix $W^*(z)W(z)$. From the equality $\lambda_1^2 \lambda_2^2 \cdots \lambda_n^2 = e^{-4l \operatorname{Im} z}$ it follows that

$$\lambda_1^2 = \frac{1}{\lambda_2^2 \cdots \lambda_n^2}\, e^{-4l \operatorname{Im} z} \geqslant \frac{1}{\lambda_n^{2(n-1)}}\, e^{-4l \operatorname{Im} z}.$$

From (3) it follows that

$$\lambda_n^2 = \|W^*(z)W(z)\| = \|W(z)\|^2 \leqslant e^{-2\sigma \operatorname{Im} z} \qquad (\operatorname{Im} z \leqslant 0)$$

and, consequently,

$$\lambda_1^2 \geqslant e^{-2[2l - \sigma(n-1)] \operatorname{Im} z} \qquad (\operatorname{Im} z \leqslant 0).$$

Introducing the notation $l_0 = 2l - \sigma(n-1)$ and taking into account that λ_1^2 is the smallest eigenvalue of the matrix $W^*(z)W(z)$, we obtain the inequality

$$W^*(z)W(z) \geqslant e^{-2l_0 \operatorname{Im} z} I \qquad (\operatorname{Im} z \leqslant 0). \tag{4}$$

Since $\sigma < \dfrac{2l}{n-1}$, we have $l_0 > 0$, and the scalar matrix function $W_0(z) = e^{il_0 z} I$ satisfies conditions 1)–4). We shall show that the matrix function $W_1(z) = W(z)W_0^{-1}(z)$ also satisfies these conditions. The fulfillment of conditions 1)–3) is obvious. To verify condition 4), consider the equality

$$W_1^*(z)W_1(z) = W_0^{*-1}(z)\, W^*(z)W(z)\, W_0^{-1}(z) =$$

$$= e^{il_0 \bar{z} I}\, W^*(z)W(z)\, e^{-il_0 z I} = e^{2l_0 \operatorname{Im} z} W^*(z)W(z),$$

whence, by virtue of (4), the inequality

$$W_1^*(z)W_1(z) \geqslant I \qquad (\operatorname{Im} z < 0)$$

follows.

Thus, $W(z)$ has the scalar divisor $W_0(z)$, and, consequently, the operator A is not unicellular.

Consider the case when $\sigma = 2l/n$. From (2) and (3) it follows that

$$\|W(z)\| = e^{-\frac{2l}{n}\operatorname{Im} z} \qquad (\operatorname{Im} z \leqslant 0).$$

Therefore, as is easy to prove,

$$\lambda_1^2 = \lambda_2^2 = ... = \lambda_n^2 = e^{-\frac{4l}{n}\operatorname{Im} z} \qquad (\operatorname{Im} z \leq 0),$$

i.e., the matrix-function $W^*(z)W(z)$ is scalar in the lower half-plane. Since $W(z) = W^{*-1}(\bar{z})$ ([1]), it follows that

$$\begin{aligned}\|W(z)\|^2 = \|W^*(z)W(z)\| &= \|W^{-1}(\bar{z})W^{*-1}(\bar{z})\| \\ = \|[W^*(\bar{z})W(\bar{z})]^{-1}\| &= e^{\frac{4l}{n}\operatorname{Im} \bar{z}} = e^{-\frac{4l}{n}\operatorname{Im} z} \qquad (\operatorname{Im} z > 0).\end{aligned}$$

Thus, the equality

$$\|W(z)\| = e^{-\frac{2l}{n}\operatorname{Im} z}$$

holds for all z. Considering the scalar matrix-function

$$W_0(z) = e^{\frac{2ilz}{n}} I,$$

we obtain that $\|W(z)W_0^{-1}(z)\| \equiv 1$, whence it follows that the matrix-function $W(z)W_0^{-1}(z)$ is identically equal to some constant matrix. From condition 2 it is clear that $W(z)W_0^{-1}(z) \equiv I$; consequently,

$$W(z) = e^{\frac{2ilz}{n}} I. \tag{5}$$

Considering the triangular model ([1]) of the operator with characteristic matrix-function (5), we obtain the following result.

Theorem 2. *If $\sigma = 2l/n$, then the space $\mathfrak{H}$ decomposes into the orthogonal sum $\mathfrak{H} = \mathfrak{H}_1 \oplus ... \oplus \mathfrak{H}_n$ of subspaces invariant with respect to the operator A, in each of which the induced operator is unicellular and has a one-dimensional imaginary component.*

Consider the multiplicative representation ([1]) of the characteristic matrix-function

$$W(z) = \int_0^l e^{2izt}\,dH(t)\left(H(t) = \int_0^t B(x)\,dx, \quad B^*(x) = B(x), \quad B(x) \geq 0, \quad \operatorname{sp} B(x) \equiv 1\right). \tag{6}$$

The idea of the proof of the following assertion belongs to Yu. L. Shmul' yan.

Lemma 3. *If the rank of the matrix-function $B(t)$ is equal to n on a set of positive measure, then the operator A is not unicellular.*

Proof. Let $m(t)$ be the lower bound of the spectrum of the matrix $B(t)$; then $m(t) > 0$ ($t \in e_0 \subset [0, l]$, $\operatorname{mes} e_0 > 0$) and

$$l_0 = \int_0^l m(t)\,dt > 0.$$

It is not difficult to prove that the scalar matrix-function

$$W_0(z) = e^{2il_0 z} I$$

is a divisor of the characteristic matrix-function $W(z)$, and, consequently, the operator A is not unicellular.

2. Consider the case $n = 2$. Comparing the criterion of M. S. Brodskii with Lemma 2 and Theorem 1, we obtain the following results.

Theorem 3. *In order that a simple Volterra dissipative operator with a two-dimensional imaginary component be unicellular, it is necessary and sufficient that the type of growth σ of its characteristic matrix-function be equal to the non-Hermitian trace $2l$ of the operator.*

Theorem 4. *In order that a simple Volterra dissipative operator with a two-dimensional imaginary component be unicellular, it is necessary and sufficient that its characteristic matrix-function of the second order have no scalar divisors.*

Theorem 5. *For any simple Volterra dissipative operator with a two-dimensional imaginary component there exists a spectral function ([5]) of rank one.*

Proof. It suffices to prove that the c. m.-f. of the operator is representable in the form (6), where the rank of the matrix-function $B(t)$ is equal to 1 almost everywhere. In the case $\sigma = 2l$ this follows from Lemma 3, since, by M. S. Brodskii' s criterion, the operator is unicellular.

If $\sigma = l$, then

$$W(z) = e^{ilzI} = e^{ilzP_2} e^{ilzP_1} = \int_0^l e^{2iz} \, dH(t),$$

where

$$P_1 + P_2 = I, \qquad P_1 P_2 = 0, \qquad H(t) = \int_0^t B(x) \, dx,$$

$$B(x) = \begin{cases} P_1, & \text{for } 0 \le t < l/2, \\ P_2, & \text{for } l/2 \le t \le l. \end{cases}$$

Let $l < \sigma < 2l$; then $W(z) = e^{il_0 zI} W_1(z)$ $(l_0 = 2l - \sigma)$. It can be shown that $W_1(z)$ has no scalar divisors and, consequently, is regular; therefore, in the multiplicative representation

$$W_1(z) = \int_0^l e^{2iz} \, dH_1(t) \left(H_1(t) = \int_0^t B_1(x) \, dx, \quad l_1 = \sigma - l \right)$$

the rank of the matrix-function $B_1(t)$ is equal to 1 almost everywhere. Defining the matrix-function $B(t)$ by the equalities

$$B(t) = \begin{cases} B_1(t), & \text{for } 0 \le t < l_1, \\ P_1, & \text{for } l_1 \le t < l_1 + \frac{1}{2} l_0, \\ P_2, & \text{for } l_1 + \frac{1}{2} l_0 \le t \le l_1 + l_0, \end{cases} \qquad (P_1 + P_2 = I, \quad P_1 P_2 = 0)$$

we obtain the obvious representation

$$W(z) = \int_0^l e^{2iz} \, dH(t) \left(H(t) = \int_0^t B(x) \, dx, \quad l = l_1 + l_0 \right).$$

Let A be a simple Volterra dissipative operator with two-dimensional imaginary component, and let $W(z)$ be its c. m.-f. of the second order. Then 3 cases are possible:

I. $\sigma = 2l$; in this case the operator A is unicellular, and the c. m.-f. $W(z)$ is regular.

II. $\sigma = l$; in this case the operator decomposes into an orthogonal sum of two unicellular operators, and the c. m.-f. $W(z)$ is scalar.

III. $l < \sigma < 2l$; in this case the operator A has two mutually orthogonal invariant subspaces, and the c. m.-f. is representable in the form $W(z) = W_0(z)W_1(z)$, where $W_1(z)$ is regular and $W_0(z)$ is scalar.

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Note: Figure translations are in progress. See original paper for figures.