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Abstract

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ON ESTIMATING THE SPECTRUM OF CERTAIN CLASSES OF LINEAR OPERATORS

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The paper considers the question of estimating the spectrum of linear operators acting in a Banach space E, semi-ordered by a cone K in the sense of M. G. Krein. Estimates of this kind for linear positive operators were obtained earlier by M. G. Krein (^{1}) and M. A. Krasnosel' skii (^{2}). For example, in (^{1}) it was shown that if K is a solid normal cone in E and $A(E \to E)$ is a linear positive $(AK \subset K)$ operator, and for some u, which is an interior element of K, $Au = \rho u$, then the spectrum of the operator A is contained in the disk $|\lambda| \leqslant \rho$. It turns out that M. G. Krein' s idea of estimating the spectrum of a linear operator from the behavior of the operator on just one interior element of a solid cone K admits an essential development.

Everywhere below it is assumed that K is a normal (^{1}) cone in E.

1. Lemma 1. Let A be a linear positive operator. Suppose that for a fixed element $f \in K$, for some λ_0,

$$A^m f \leqslant \lambda_0 f. \tag{1}$$

Then, for $|\lambda| > \sqrt[m]{\lambda_0}$, the equation

$$\lambda x = Ax + f \tag{2}$$

has at least one solution x^*:

$$x^* = \sum_{n=0}^{\infty} \frac{A^n f}{\lambda^{n+1}}.$$

2. Let g be a fixed element of the cone K. Consider the set E_g of those elements $x \in E$ for which, for some $a \geqslant 0$, the inequalities

$$-ag \leqslant x \leqslant ag \tag{3}$$

hold.

Obviously, E_g is a linear manifold in E. For the elements of this linear manifold one may introduce a new norm $\| \cdot \|_g$, setting $\|x\|_g = \inf\{a\}$, where the infimum is taken over the set of those values of a for which the inequalities (3) hold (see (^{2})). With the aid of Lemma 1 the following is proved.

Theorem 1. Let a positive linear operator A satisfy the inequality

$$A^m g \leqslant \lambda_0 g, \tag{4}$$

where g is some fixed element of K.

Then, for $|\lambda| > \sqrt[m]{\lambda_0}$, equation (2) has, for every $f \in E_g$, a solution x^*, unique in E_g, and

$$x^* = R_\lambda f = \sum_{n=0}^{\infty} \frac{A^n f}{\lambda^{n+1}}.$$

Corollary. If g is an interior element of the cone K and $A^m g \leqslant \lambda_0 g$, then the spectrum of the linear positive operator A is contained in the disk

$$|\lambda| \leqslant \sqrt[m]{\lambda_0}.$$

3. In some cases the corollary to Theorem 1 makes it possible to obtain a more precise estimate for the radius ρ_0 of the disk in which the spectrum of the operator A lies, in comparison with the estimate given by the norm of the operator. We give the corresponding example.

Let

$$A(\varphi(t)) = \int_0^1 K(t,s) a(s)\, ds, \tag{5}$$

where the continuous kernel $K(t,s)$ has the form

$$K(t,s) = ts^2 K_1(t,s),$$

and the function $K_1(t,s)$, continuous in the square, is such that

$$0 \leqslant K_1(t,s) \leqslant M \qquad (0 \leqslant t, s \leqslant 1).$$

It is not difficult to see that then, in the space $C(0,1)$, $\|A\| = M/3$. Under the assumptions made, the operator A is positive with respect to the cone of nonnegative functions of this space. The functions $\varphi_n(t) = (1 + nt)$ $(0 \leqslant t \leqslant 1)$ are interior elements of this cone. Obviously,

$$A\varphi_n(t) \leqslant t \int_0^1 Ms^2(1+ns)\,ds = M\left(\frac{1}{3}+\frac{n}{4}\right)t,$$

whence it follows that

$$A\varphi_n \leqslant M\left[\frac{1}{3(n+1)}+\frac{n}{4(n+1)}\right]\varphi_n, \tag{6}$$

and therefore the spectrum of the operator A lies in the disk $|\lambda| \leqslant M/4$. Note that, for kernels $K(t,s)$ of the form indicated in this example, the estimate $|\lambda| \leqslant M/4$ for the spectrum of the operator cannot be improved: if $K_1(t,s) = M$, then the operator (5) has the eigenvalue $\lambda_0 = M/4$.

4. We shall call the operator A g-bounded above ($g \in K,\ g \neq \theta$) if for every $x \in K$ there exist a natural number m and a number $a > 0$ such that

$$A^m x \leqslant ag\ (^2).$$

Theorem 2. Let K be a reproducing cone in E, and let A be a linear positive operator g-bounded above, where

$$A^s g \leqslant \lambda_0 g. \tag{7}$$

Then the spectrum of the operator A is contained in the disk

$$|\lambda| \leqslant \sqrt[s]{\lambda_0}. \tag{8}$$

Proof. Suppose first that $f \in K$. By virtue of the g-boundedness of the operator A above, there is a natural number m such that

$$\theta \leqslant A^m f \leqslant ag \qquad (a > 0).$$

Applying to this inequality the positive operator A^{ps} and taking (7) into account, we obtain

$$\theta \ll A^{m+ps} f \ll a\lambda_0^p g \qquad (p = 1, 2, \ldots). \tag{9}$$

Let $n > m + s$; then n can be represented in the form

$$n = (m + p_1 s) + p_2, \tag{10}$$

where p_1 and p_2 are integers, and $0 \leq p_2 < s$. Denote

$$\max\{\|Ag\|,\ \|A^2 g\|, \dots, \|A^s g\|\} = M. \tag{11}$$

If n satisfies (10), then

$$\|A^n f\| = \|A^{(m+p_1 s)+p_2} f\| = \|A^{p_2} A^{m+p_1 s} f\|.$$

But

$$\theta \ll A^n f \ll A^{p_2} A^{m+p_1 s} f \ll a\lambda_0^{p_1} A^{p_2} g.$$

Therefore, by virtue of the normality of the cone K,

$$\|A^n f\| \leq Ca\lambda_0^{p_1} \|A^{p_2} g\| \leq C_1 \lambda_0^{p_1},$$

where C is the constant entering into the definition of the semimonotonicity of the norm $(^2)$, and

$$C_1 = CaM.$$

Next consider the numerical positive-term series

$$\sum_{n=0}^{\infty} \frac{\|A^n f\|}{|\lambda|^{n+1}}. \tag{12}$$

For $|\lambda| > \sqrt[s]{\lambda_0}$, from the following estimates it is not difficult to see that it converges:

$$\sum_{n=0}^{\infty} \frac{\|A^n f\|}{|\lambda|^{n+1}} = \sum_{n=0}^{m+s-1} \frac{\|A^n f\|}{|\lambda|^{n+1}} + \sum_{p_1=1}^{\infty} \sum_{p_2=0}^{s-1} \frac{\|A^{m+p_1 s+p_2} f\|}{|\lambda|^{m+p_1 s+p_2+1}} \leq$$

$$\leq \sum_{n=0}^{m+s-1} \frac{\|A^n f\|}{|\lambda|^{n+1}} + \sum_{p_2=0}^{s-1} \frac{C_1}{|\lambda|^{m+p_2+1}} \sum_{p_1=1}^{\infty} \left(\frac{\lambda_0}{|\lambda|^s}\right)^{p_1}.$$

From the convergence of the series (12) follows the strong convergence of the series

$$R_\lambda f \equiv \sum_{n=0}^{\infty} \frac{A^n f}{\lambda^{n+1}}. \tag{13}$$

Further, it is obvious that

$$\lambda R_\lambda f = AR_\lambda f + f,$$

i.e. $R_\lambda f$ is a solution of equation (2). We now show that the series (12) converges for every $f \in E$.

Let $f \in E$; then

$$f = f_1 - f_2 \qquad (f_1, f_2 \in K),$$

and, consequently, by virtue of the convergence, for $|\lambda| > \sqrt[s]{\lambda_0}$, of the series

$$\sum_{n=0}^{\infty} \frac{\|A^n f_1\|}{|\lambda|^{n+1}}, \qquad \sum_{n=0}^{\infty} \frac{\|A^n f_2\|}{|\lambda|^{n+1}},$$

there follows the convergence of the series (12), and along with it the strong convergence of the series

(13). Thus, it has been shown that for $|\lambda| > \sqrt[s]{\lambda_0}$ and any $f \in E$ the series (13) is a solution of equation (2). The uniqueness of this solution is obvious. The theorem is proved.

Theorem 3. Let A be a linear bounded positive operator such that for each $x \in E$ there exist natural numbers $m = m(x)$ and $r = r(x)$ such that, for some $a = a(x) > 0$ and $b = b(x) > 0$, the relations

$$A^m x \leq ag, \quad -bg \leq A^r x, \quad A^s g \leq \lambda_0 g$$

hold.

Then the spectrum of the operator A is contained in the disk

$$|\lambda| \leq \sqrt[s]{\lambda_0}.$$

5. We now dispense with the requirement that the operator A be positive. Instead, we shall assume that there exist positive operators A_1 and A_2 such that, for every $x \in K$, the relation

$$-A_1 x \leq A^p x \leq A_2 x$$

holds.

Theorem 4. Let the positive operators A_1 and A_2 satisfy, at some element $g \neq \theta$ of the normal cone K, the inequalities

$$A_1 g \leq \lambda_1 g, \quad A_2 g \leq \lambda_2 g.$$

Then equation (2), for all $f \in E_g$, has a unique solution in E_g whenever $|\lambda| > \sqrt[p]{\lambda_1 + \lambda_2}$.

If, moreover, the operators A_1, A_2, and $A_1 + A_2$ are g-bounded from above, and the cone K is reproducing, then the spectrum of the operator A lies in the disk

$$|\lambda| \leq \sqrt[p]{\lambda_1 + \lambda_2}.$$

The author, jointly with A. R. Esayan, has established that in the case $A_1 = A_2$ the last estimate can be improved: the spectrum of the operator A lies in the disk

$$|\lambda| \leq \sqrt[p]{\max(\lambda_1, \lambda_2)}.$$

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REFERENCES

1. M. G. Krein, M. A. Rutman, *UMN*, 3, issue 1 (23) (1948); 2. M. A. Krasnosel' skii, *Positive Solutions of Operator Equations*, Moscow, 1962.

Note: Figure translations are in progress. See original paper for figures.

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