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Abstract

Full Text

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Completely FN -Factorizable Groups

(Presented by Academician A. I. Mal' tsev on 6 XII 1963)

A subgroup \mathfrak{A} of a group \mathfrak{G} is called F -**complemented** in \mathfrak{G} if there exists in \mathfrak{G} a subgroup \mathfrak{B} such that $\mathfrak{A}\mathfrak{B} = \mathfrak{G}$ and the intersection $\mathfrak{A} \cap \mathfrak{B}$ is finite. If the intersection $\mathfrak{A} \cap \mathfrak{B}$ lies in a finite normal divisor of the group \mathfrak{B} , then the subgroup \mathfrak{A} is called FN -**complemented**; in particular, if $\mathfrak{A} \cap \mathfrak{B} = 1$, then the subgroup \mathfrak{A} is called simply **complemented** in \mathfrak{G} .

A group \mathfrak{G} is called **completely F -factorizable**, respectively **completely FN -factorizable**, **completely factorizable**, if each of its subgroups is F -complemented, respectively FN -complemented, simply complemented in the group \mathfrak{G} .

Completely factorizable groups were completely studied by N. V. Chernikova (Baeva) in papers ^(1,2). Groups with various sorts of systems of complemented subgroups and, in particular, with complemented abelian subgroups were studied by S. N. Chernikov (see ⁽³⁾).

In the present article, which gives a solution to a question posed by S. N. Chernikov on the structure of groups with abelian FN -complemented subgroups, completely FN -factorizable groups are mainly considered. However, most of the results are formulated for completely F -factorizable groups.

Let us note here that the study of groups with F -complemented subgroups is connected with the still unsolved question of the existence of noncommutative O. Yu. Schmidt groups, i.e., groups all of whose proper subgroups are finite (these groups are, evidently, completely F -factorizable); therefore it is not possible to describe completely the structure of completely F -factorizable groups.

1. Every completely F -factorizable group is periodic. This follows from the proposition:

Every subgroup of every factor group of a completely F -factorizable (completely FN -factorizable) group is completely F -factorizable (completely FN -factorizable).

It is unknown, however, whether an arbitrary completely F -factorizable group is locally finite; this is unknown even for completely F -factorizable groups that are O. Yu. Schmidt groups. If, however, a completely F -factorizable group is completely FN -factorizable, then it is locally finite by virtue of the following proposition.

Lemma 1. *Every group with FN -complemented abelian subgroups is an extension of a locally normal group by means of a completely factorizable group.*

In the proof of this lemma one uses a result from paper ⁽⁴⁾ (see therein Lemma 9) and the proposition:

Every locally normal group with FN -complemented abelian subgroups and with complemented center can be embedded in the direct product \mathfrak{R} of a certain extremal group and a certain completely factorizable group.

An **extremal group** is a finite extension of an abelian group satisfying the minimality condition for subgroups.

The direct product \mathfrak{R} , in which a locally normal group with FN -complemented abelian subgroups is embedded, is a completely FN -factorizable group. This follows from the propositions:

1. *If a locally normal group is extremal, then it is completely FN -factorizable.*
2. *The direct product of a finite number of completely F -factorizable groups is a completely F -factorizable group.*

For arbitrary groups with FN -complemented abelian subgroups the following proposition is valid:

Every group \mathfrak{G} with FN -complemented abelian subgroups is representable as the product of a subgroup that is an extension of a finite normal divisor of the group \mathfrak{G} by a completely factorizable group, and of a normal divisor every element of which lies in a finite class of conjugate elements of the group \mathfrak{G} .

Hence, and from the results cited above, there follows a generalization of one of the results of paper ⁽³⁾, namely:

Theorem 1. *Every group with FN -complemented abelian subgroups is completely FN -factorizable.*

This theorem establishes the equivalence of the condition that all abelian subgroups of a certain group be FN -complemented and the condition that it be completely FN -factorizable.

§ 2. The theorems below give a description of the structure of completely FN -factorizable groups.

Theorem 2. *A group \mathfrak{G} is completely FN -factorizable if and only if it contains such a finite normal divisor \mathfrak{N} that the factor group $\mathfrak{G}/\mathfrak{N}$ decomposes into the semidirect product of a completely factorizable group $\mathfrak{A}/\mathfrak{N}$ and a normal divisor $\mathfrak{R}/\mathfrak{N}$, which is representable as the direct product of a completely factorizable group and a complete abelian group with the minimality condition for subgroups, and all abelian subgroups of which are FN -complemented in $\mathfrak{G}/\mathfrak{N}$.*

The author has an example showing that the last requirement in this theorem cannot be omitted.

The proof of this theorem is obtained with the help of Propositions 1 and 2 noted above and the proposition:

An extension of a finite group by a completely FN-factorizable (completely F-factorizable) group is a completely FN-factorizable (completely F-factorizable) group. In the case of completely factorizable groups this assertion does not hold.

Theorem 3. *Every completely FN-factorizable group \mathfrak{G} is representable as the product of a finite normal divisor coinciding with its commutator subgroup, and a solvable subgroup.*

The proof of Theorem 3 follows from Theorem 2 and the following lemma.

Lemma 2. *Every completely F-factorizable group is representable as the product of its commutator subgroup and some nilpotent subgroup.*

This lemma can easily be proved with the help of the proposition:

3. *A completely F-factorizable group is nilpotent if and only if its commutator subgroup has no F-complements other than the group itself.*

The latter is proved with the help of the following two propositions, which are also of some independent interest.

The commutator subgroup of a group \mathfrak{G} representable as the product of two of its subgroups \mathfrak{A} and \mathfrak{B} is the product of their mutual commutator $[\mathfrak{A}, \mathfrak{B}]$ and the subgroup $\{\mathfrak{A}', \mathfrak{B}'\}$ generated by the commutators \mathfrak{A}' and \mathfrak{B}' of the subgroups \mathfrak{A} and \mathfrak{B} .

If a group \mathfrak{G} coincides with the product of its commutator subgroup \mathfrak{G}' and some normal divisor \mathfrak{N} , then it is generated by the subgroup \mathfrak{N} and by any element of the commutator chain having a natural number.

Remark. If here the normal divisor \mathfrak{N} is finite, then \mathfrak{G} is generated by this normal divisor and by any element of the commutator chain. Hence it follows that the factor group of an infinite locally normal KK -group by its commutator subgroup is infinite. This proposition is used in proving the propositions of the following § 3.

§ 3. In this section propositions are presented describing the structure of solvable completely FN-factorizable groups.

From Lemma 2 and Proposition 3 there follows the validity of the following proposition:

Every solvable completely FN-factorizable group is representable as the product of some finite system of its nilpotent

subgroups $\mathfrak{A}_0, \mathfrak{A}_1, \dots, \mathfrak{A}_k$ satisfying the following conditions: a) the subgroup \mathfrak{B}_i , $0 < i \leq k$, generated by all subgroups \mathfrak{A}_j , $j < i$, is a normal divisor of the group \mathfrak{G} ; b) for all \mathfrak{B}_i , $0 < i \leq k$, the intersection $\mathfrak{B}_i \cap \mathfrak{A}_i$ is finite and the factor group $\mathfrak{A}_i / \mathfrak{B}_i \cap \mathfrak{A}_i$ is abelian; c) the subgroup \mathfrak{B}_{k-3} is finite.

Remark. An analogous proposition holds for completely F -factorizable groups; unlike the proposition formulated here, in it the group \mathfrak{B}_{k-3} need not be finite.

Theorem 4. *A nilpotent group \mathfrak{G} is completely F -factorizable if and only if it can be represented as the product of two of its normal divisors, one of which is an extremal subgroup and the other a periodic subgroup whose element orders do not contain squares of prime divisors and whose commutator subgroup coincides with some finite subgroup of the center of the group \mathfrak{G} .*

Corollary. *An abelian group is completely F -factorizable if and only if it can be represented as the direct product of a completely factorizable subgroup and a subgroup satisfying the minimality condition for subgroups.*

In the proof of Theorem 4, results from the papers of S. N. Chernikov ^(6,7), results of the paper of O. N. Golovin ⁽⁵⁾, Proposition 3, and the following proposition are used:

If a group with finite commutator subgroup can be represented as the product of two completely F -factorizable groups, then it is completely F -factorizable.

It follows without difficulty from Theorem 4 that the class of completely F -factorizable nilpotent groups coincides with the class of completely FN -factorizable nilpotent groups.

p. 4. If a completely F -factorizable group has an upper central series, i.e. is a ZA -group, then the following holds.

Theorem 5. *If a completely F -factorizable ZA -group \mathfrak{G} is not nilpotent, then it contains an invariant abelian 2-group \mathfrak{P} , which can be represented as the direct product of invariant in \mathfrak{G} quasi-classical groups and whose F -complements in \mathfrak{G} are nilpotent groups.*

The proof of Theorem 5 is obtained with the help of the results of the paper ⁽⁸⁾, Proposition 3, and the following propositions:

- a) *Every completely F -factorizable group can be represented as the direct product of an abelian completely factorizable group and a group whose center satisfies the minimality condition for subgroups.*
- b) *If the Frattini subgroup of a group \mathfrak{G} coincides with its identity subgroup, then the group \mathfrak{G} can be represented as the direct product of its center and some subgroup without center.*

It follows from these propositions that the commutator subgroup of every ZA -group is contained in its Frattini subgroup.

Remark. a) Since every completely FN -factorizable periodic S -group is a ZA -group (which follows from Theorem 2 and A. P. Ditzman's theorem on the center of a p -group), its structure is described by Theorems 4 and 5; b) it follows from Theorem 5 that the class of completely F -factorizable ZA -groups coincides with the class of completely FN -factorizable ZA -groups.

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