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Abstract

Full Text

MATHEMATICS

A. K. KHARADZE

**ON THE REPRESENTATION OF ULTRAS-
PHERICAL POLYNOMIALS IN THE FORM
OF A DIFFERENTIAL OPERATOR CON-
TAINING THE GENERATING FUNCTION
OF THESE POLYNOMIALS**

(Presented by Academician P. S. Aleksandrov on 6 IV 1964)

Formulas of Rodrigues type give a representation of classical polynomials in the form of a differential operator whose composition includes the weight function. But it is known that the Hermite polynomial $H_n(x)$ also has another representation—in the form of a linear differential operator containing the generating function. Thus, L. Poli showed ⁽¹⁾ that, if $y = e^{x^2-t^2/2} =$

$$= \sum_0^{\infty} H_n(x) \frac{t^n}{n!},$$

then the equality holds

$$yH_n(x) = \delta_n[y],$$

where

$$\delta_n[y] \equiv y^{(n)} + \binom{n}{1} ty^{(n-1)} + \dots + \binom{n}{n-1} t^{n-1} y' + t^n y, \quad y^{(k)} \equiv \frac{d^k y}{dt^k}.$$

From this, as a consequence, the general solution of the equation $\delta_n[y] = 0$ is obtained directly in the form $y = C_1 y_1 + C_2 y_2 + \dots + C_n y_n$, where $y_i = e^{x_i t - t^2/2}$, x_i are the roots of the equation $H_n(x) = 0$.*

In connection with this, it is of interest to consider also other sequences of polynomials $\{\varphi_n(x)\}$ according to the following scheme: let $y = f(x, t) =$

$$= \sum_0^{\infty} \varphi_n(x) t^n$$

and $\varphi_n(x) = \Omega[y]$; then the functions $f(x_i, t)$ will be solutions of the equation $\Omega[y] = 0$, where x_i are the roots of the polynomial $\varphi_n(x)$.

In the present note a representation is given of the ultraspherical polynomials $P_n^{(\lambda)}(x)$ in the form of a nonlinear differential operator containing the generating function of these polynomials, and at the same time particular solutions are given of the corresponding differential equation, into whose expressions the roots of the mentioned polynomials enter as values of a parameter.

Thus, let

$$y \equiv \frac{1}{(1 - 2xt + t^2)^\lambda} = \sum_0^\infty P_n^{(\lambda)}(x)t^n, \quad P_0^{(\lambda)} = 1, \quad P_1^{(\lambda)} = 2\lambda x.$$

Consider the expression

$$\begin{aligned} F(z, t, y) &\equiv P_n^{(\lambda)}(z) + \binom{\omega}{1} t y^{1/2\lambda} P_{n-1}^{(\lambda)}(z) + \dots \\ &\dots + \binom{\omega}{n-1} t^{n-1} y^{(n-1)/2\lambda} P_1^{(\lambda)}(z) + \binom{\omega}{n} t^n y^{n/2\lambda} P_0^{(\lambda)}(z), \end{aligned} \quad (1)$$

where $z = y^{1/2\lambda}(x - t)$, $\omega = n + 2\lambda - 1$, and first of all let us prove the validity of the equality

$$F(z, t, y) = y^{n/2\lambda} P_n^{(\lambda)}(x).$$

* The equation $\delta_n[y] = 0$ and its generalizations were also considered in (2-4).

Taking into account that $dy/dt = y^{1/2\lambda+1} 2\lambda z$, $dz/dt = y^{1/2\lambda}(z^2 - 1)$, as a result of differentiating (1) with respect to t we shall have

$$\frac{d}{dt} F(z, t, y) = \frac{\partial F}{\partial z} y^{1/2\lambda}(z^2 - 1) + \frac{\partial F}{\partial t} + \frac{\partial F}{\partial y} y^{1/2\lambda+1} 2\lambda z. \quad (2)$$

Since the sequence $\{P_n^{(\lambda)}(z)\}$ satisfies the relation

$$(z^2 - 1) \frac{d}{dz} P_n^{(\lambda)}(z) = nz P_n^{(\lambda)}(z) - (n + 2\lambda - 1) P_{n-1}^{(\lambda)}(z),$$

then, carrying out the required calculations in the right-hand side of (1), we obtain

$$\frac{d}{dt}F(z, t, y) = nzy^{1/2\lambda}F(z, t, y). \quad (3)$$

On the other hand, since the right-hand side of (1) is a polynomial of degree n with respect to z , equality (1) may be written in the form

$$F(z, t, y) = y^{n/2\lambda}\Phi. \quad (4)$$

It is easy to establish that Φ does not depend on t . Indeed, differentiation with respect to t gives

$$\frac{d}{dt}F(z, t, y) = nzy^{(n+1)/2\lambda}\Phi + y^{n/2\lambda}\frac{d}{dt}\Phi$$

or, according to (4),

$$\frac{d}{dt}F(z, t, y) = nzy^{1/2\lambda}F(z, t, y) + y^{n/2\lambda}\frac{d}{dt}\Phi,$$

and therefore, on the basis of (3), $d\Phi/dt = 0$. Thus, since Φ does not depend on t , we can find its expression by putting $t = 0$ in equality (4); but then $y^{1/2\lambda} = 1$, $z = x$, and hence we finally obtain

$$y^{n/2\lambda}P_n^{(\lambda)}(x) = P_n^{(\lambda)}(z) + \binom{\omega}{1}ty^{1/2\lambda}P_{n-1}^{(\lambda)}(z) + \dots + \binom{\omega}{n}t^{ny^{n/2\lambda}}P_0^{(\lambda)}(z). \quad (I)$$

Now let us show that if $y \equiv (1 - 2xt + t^2)^{-\lambda}$, then

$$y^{(n)} \equiv \frac{d^ny}{dt^n} = n!y^{n/2\lambda+1}P_n^{(\lambda)}(z), \quad (5)$$

where z has the preceding meaning. Indeed, for $n = 1$ this is obvious, for $dy/dt = y^{1/2\lambda+1}P_1^{(\lambda)}(z)$. If formula (5) is true for some n , then, differentiating it with respect to t , we obtain

$$\begin{aligned} \frac{d^{n+1}y}{dt^{n+1}} &= n! \left(\frac{n}{2\lambda} + 1 \right) y^{(n+1)/2\lambda+1} 2\lambda z P_n^{(\lambda)}(z) + n! y^{n/2\lambda+1} \frac{d}{dz} P_n^{(\lambda)}(z) y^{1/2\lambda} (z^2 - 1) = \\ &= n! y^{(n+1)/2\lambda+1} \left[(n + 2\lambda) z P_n^{(\lambda)}(z) + (z^2 - 1) \frac{d}{dz} P_n^{(\lambda)}(z) \right] = \\ &= (n + 1)! y^{(n+1)/2\lambda+1} P_{n+1}^{(\lambda)}(z). \end{aligned}$$

Thus, (5) is valid for every natural n . Returning to equality (I), we substitute instead of $P_k^{(\lambda)}(z)$ ($k = 0, 1, \dots, n$) their expressions from (5), which after simple transformations gives

$$y^{(n+\lambda)/\lambda} P_n^{(\lambda)}(x) = D_n^{(\lambda)}[y] \equiv \frac{y^{(n)}}{n!} + \binom{\omega}{1} t y^{1/\lambda} \frac{y^{(n-1)}}{(n-1)!} + \dots$$

$$\dots + \binom{\omega}{n-1} t^{n-1} y^{(n-1)/\lambda} \frac{y'}{1!} + \binom{\omega}{n} t^n y^{n/\lambda+1}, \quad (\text{II})$$

where $y \equiv (1 - 2xt + t^2)^{-\lambda}$.

It is now clear that the functions $(1 - 2x_{it} + t^2)^{-\lambda}$ will be solutions of the equation $D_n^{(\lambda)}[y] = 0$, where x_i are the roots of the polynomial $P_n^{(\lambda)}(x)$.

Equalities (I) and (II) take the simplest form for $\lambda = 1/2$, which corresponds to the Legendre polynomials $P_n(x)$. In this case we have

$$y^n P_n(x) = P_n(z) + \binom{n}{1} t y P_{n-1}(z) + \dots + \binom{n}{n-1} t^{n-1} y^{n-1} P_1(z) + t^n y^n P_0(z), \quad (\text{I}^*)$$

$$y^{2n+1} P_n(x) = \frac{y^{(n)}}{n!} + \binom{n}{1} t y^2 \frac{y^{(n-1)}}{(n-1)!} + \dots + \binom{n}{n-1} t^{n-1} y^{2(n-1)} \frac{y'}{1!} + t^n y^{2n+1}, \quad (\text{II}^*)$$

where $y = 1/\sqrt{1 - 2xt + t^2}$, $z = y(x - t)$.

It should be noted that the analysis given above is also applicable in the general case of Jacobi polynomials; however, the effective notation of relations analogous to (I) and (II) is connected with algebraic complications.

Tbilisi
State University

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REFERENCES

- ¹ L. Poli, *Mathesis*, **63**, No. 9–10, 319 (1954).
- ² F. Moretti, *Riv. di Mat. d. Univ. di Parma*, **1**, 471 (1950).
- ³ L. Godeaux, *Mathesis*, **64**, No. 3–5, 81 (1955).
- ⁴ A. Kharadze, *Tr. Tbilissk. gos. univ.*, **76**, 43 (1959).

Note: Figure translations are in progress. See original paper for figures.

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