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Abstract

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ON THE THEORY OF INTEGRAL REPRESENTATIONS OF FINITE GROUPS

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I. Let $G = \langle a \rangle$ be a cyclic group of order mp^α ($(m,p) = 1$; p prime). In ([1]) (see also ([2])) it is shown that if $\alpha \geqslant 3$, then the group G has indecomposable representations of arbitrarily large degree over the rings of rational integers Z, p-adic integers J_p, and p-integral rational numbers R_p. If $\alpha = 1$, then, as established in ([3]), all indecomposable Z-representations (R_p-representations) of the group G are realized in ideals of the group ring ZG (R_{pG}).

Let $d_1, \dots, d_s$ be all the distinct positive divisors of the number mp^2, and let ε_i be a primitive d_i-th root of unity ($i = 1, \dots, s$). The irreducible R_p-representations of the group $G = \langle a \rangle$ of order mp^2 are exhausted by the representations

$$\Gamma_1, \dots, \Gamma_s, \quad \text{where } \Gamma_i: \ a \to \tilde{\varepsilon}_i,$$

and $\tilde{\varepsilon}_i$ is the operator of multiplication by ε_i in the ring $M_i = R_p[\varepsilon_i]$ ($i = 1, \dots, s$).

If

$$\frac{d_i}{d_j} \neq p^\gamma \quad (\gamma \text{ an integer}),$$

then for the G–R_p-modules M_i and M_j the condition

$$\mathrm{Ext}^1(M_i, M_j) = 0$$

holds. Therefore the indecomposable R_p-representations of the group G can contain only irreducible diagonal components of the form:

$$\Gamma^{(1)} = \widetilde{\eta\xi}; \qquad \Gamma^{(2)} = \widetilde{\eta\varepsilon}; \qquad \Gamma^{(3)} = \tilde{\eta},$$

where η is an arbitrary but fixed root of unity of degree $t \mid m$; ξ and ε are, respectively, primitive roots of unity of degrees p^2 and p.

We shall say that an indecomposable R_p-representation Γ of the group G has type (α, β, γ), if the decomposition of this representation over the field of rational numbers has the form

$$\Gamma = \alpha\Gamma^{(1)} + \beta\Gamma^{(2)} + \gamma\Gamma^{(3)}.$$

For an arbitrary R_p-representation Γ of the group G, we shall denote by $n(\Gamma)$ the number of irreducible rational components of the representation Γ, and by $m(\Gamma)$ the maximum number of mutually equivalent irreducible rational components of this representation.

It is shown in ([1]) that the types of all indecomposable $R_p(J_p)$-representations of a cyclic group of order p^2 $(p \neq 2)$ are exhausted by the vectors $(1, 0, 0)$; $(0, 1, 0)$; $(0, 0, 1)$; $(1, 0, 1)$; $(1, 1, 0)$; $(0, 1, 1)$; $(1, 1, 1)$; $(1, 1, 2)$.

In the present note the types of all indecomposable R_p-representations of any group of order mp^2 are determined. In the second part of the paper a theorem is given proving the existence of infinite series of indecomposable Z (R_p, J_p)-representations of a finite group all of whose irreducible rational components are mutually equivalent.

Theorem 1. Let G be a cyclic group of order mp^2 $((m, p) = 1)$, and let δ be the exponent to which the number p belongs modulo m.

If $p \neq 2$, then the types of indecomposable R_p-representations of the group G for $\delta = \varphi(m)$ (φ is Euler' s function) are exhausted by the vectors:

$$\begin{aligned} &e_1 = (1, 0, 0); \quad e_2 = (0, 1, 0); \quad e_3 = (0, 0, 1); \quad e_4 = (1, 1, 0); \\ &e_5 = (0, 1, 1); \quad e_6 = (1, 0, 1); \quad e_7 = (1, 1, 1); \quad e_8 = (1, 1, 2), \end{aligned} \tag{1}$$

for $\delta = \varphi(m)/2$, by the vectors (1) and by the vectors:

$$\begin{gathered} (2, 1, 1); \quad (1, 2, 1); \quad (2, 1, 2); \quad (1, 2, 2); \quad (2, 2, 2); \quad (2, 3, 3); \\ (3, 2, 3); \quad (3, 2, 2); \quad (2, 3, 2), \end{gathered} \tag{2}$$

and for $\varphi(m)/\delta \geq 3$, by the vectors (1), (2), and

$$(3, 1, 2); \quad (1, 3, 2); \quad (4, 2, 2); \quad (2, 4, 2); \quad (2, 4, 4); \quad (4, 2, 4).$$

For $p = 2$ the following types arise:

$$\begin{gathered} \mathfrak{M}_1: \quad (1, 0, 0); \quad (0, 1, 0); \quad (0, 0, 1); \quad (1, 1, 0); \quad (1, 0, 1); \quad (0, 1, 1); \\ (1, 1, 1) \qquad (\delta = \varphi(m)), \end{gathered}$$

$$\mathfrak{M}_1 \cup \{(2,1,1),(1,2,1),(1,1,2),(1,2,2),(2,1,2),(2,2,2)\} \qquad (\delta \geq \varphi(m)/2).$$

Thus, every R_p-representation Γ of the group G is decomposable if at least one of the following conditions is satisfied:

$$n(\Gamma) > 10; \qquad m(\Gamma) > 4 \quad (p \neq 2),$$

$$n(\Gamma) > 6; \qquad m(\Gamma) > 2 \quad (p = 2).$$

Let us note that from Theorem 1 it follows, in particular, that the cyclic group of order mp^2 $((m,p)=1)$ for $p \neq 2$, $\varphi(m)/\delta \geq 3$, has indecomposable R_p-representations with $1,2,3,4,5,6,7,8$, and 10 rationally irreducible components.

The proof of Theorem 1 is based on the following propositions.

Lemma 1. Let $S = \{\alpha_1 e_1 + \dots + \alpha_8 e_8\}$ be the set of all possible linear combinations of the vectors (1) with nonnegative integer coefficients α_i $(i = 1, \dots, 8)$, not all zero simultaneously, and let $x = (\alpha, \beta, \gamma) \in S$ $(\alpha, \beta, \gamma > 0)$.

If $\alpha, \beta > \gamma$, then $x - (1,1,0) \in S$. If $\alpha, \beta < \gamma$, then $x - (1,1,2) \in S$. If $\alpha = \beta = \gamma \geq 2$, then $x - (2,2,2) \in S$.

If $\alpha > \beta = \gamma$, then $x - (4,2,2) \in S$ for $\alpha - \beta \geq 3$, $\beta \geq 2$; if $\alpha > \beta = \gamma$ and $\alpha - \beta = 2$, then $x - (2,2,2) \in S$ for $\beta \geq 4$, $x - (2,1,1) \in S$ for $\beta = 3$; if $\alpha > \beta = \gamma$ and $\alpha - \beta = 1$, then $x - (2,1,1) \in S$ for $\beta = 2m+1$ and $m \geq 1$, $x - (2,2,2) \in S$ for $\beta = 2m$ and $m \geq 2$.

If $\alpha < \beta = \gamma$ and $\beta - \alpha \geq 2$, then $x - (1,2,2) \in S$ for $\alpha = 2m+1$ and $x - (2,4,4) \in S$ for $\alpha = 2m$ and $m \geq 2$; if $\alpha < \beta = \gamma$ and $\beta - \alpha = 1$, then $x - (2,2,2) \in S$ for $\beta = 2m > 2$ and $x - (2,3,3) \in S$ for $\beta = 2m+1 > 3$.

If $\alpha < \gamma < \beta$, then $x - (1,3,2) \in S$ for $\beta - \alpha > 2$; if $\alpha < \gamma < \beta$ and $\beta - \alpha = 2$, then $x - (2,2,2) \in S$ for $\alpha > 2$ and $x - (1,2,1) \in S$ for $\alpha = 2$.

Lemma 2. A J_p-representation of a finite group G, equivalent over the field of p-adic numbers to a representation over the field of rational numbers, is J_p-equivalent to a representation over R_p.

Lemma 2, as applied to a cyclic group G, was used in ([4]) to prove the nonuniqueness of the decomposition of R_p-representations of the group G into a sum of indecomposable R_p-components. The general case of the lemma was proved by Heller ([4]).

Theorem 2. Let the group G be represented in the form of a semidirect product $G = H \cdot F$, where $H = (a)$ is a cyclic group of order p^α $(1 \leq \alpha \leq 2)$, and F is a group of order m $((m,p)=1)$. Let $M_1, \dots, M_t$ $(t = 4p+1$, if $\alpha = 2$, and $t = 3$, if $\alpha = 1)$ be a complete system of indecompodecomposable pairwise nonisomorphic H-J_p-modules, and $M_i^* = g_1 M_i + \dots + g_{mM} i$ ($g_1, \dots, g_m$ are elements of the group F) is the G-J_p-module induced by the H-J_p-module M_i. Let $e_1, \dots, e_r$ be the minimal idempotents of the group ring J_{pF}, to which correspond irreducible pairwise nonequivalent representations of the group F over the ring J_p (the number r is equal to the number of irreducible representations of the group F over the prime field of characteristic p). Then a complete system of indecomposable pairwise nonisomorphic G-J_p-modules is exhausted by the modules $(J_{pG}) e_{jM} i$ ($j = 1, \dots, r;\ i = 1, \dots, t$).

II. For a cyclic group G of order p^a ($p \neq 2$), with $a > 2$, in (1) a series of indecomposable Z-representations of arbitrarily large degree has been given, in which only 3 rationally nonequivalent irreducible representations of the group G occur. Other series for a cyclic group with 4 irreducible rational nonequivalent representations were constructed in (5).

On the other hand, for a finite abelian group there exists only a finite number of indecomposable Z-representations with two distinct irreducible rational components (6).

Theorem 3. Let

$$G: \quad a^p = 1; \quad b^p = 1; \quad c^p = 1; \quad ab = ba; \quad ac = ca; \quad c^{-1} bc = ab$$

($p \neq 2$; p prime). If $p > 3$, then there exist indecomposable $Z(J_p, R_p)$-representations of the group G of arbitrarily large degree, all irreducible rational components of which are equivalent to one and the same irreducible rational representation of this group of degree $\varphi(p^2)$.

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