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Abstract

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MATHEMATICS

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PERTURBATION OF SOLUTIONS AND PERTURBATION OF EIGENVALUES AND EIGENFUNCTIONS OF SECOND-ORDER ELLIPTIC EQUATIONS UNDER PERTURBATION OF THE BOUNDARY

(Presented by Academician I. G. Petrovskii on 27 II 1964)

In the papers (^{2}, ^{3}), using the methods of the paper (^{1}), we considered questions concerning the perturbation of solutions and the perturbation of eigenvalues and eigenfunctions of ordinary second-order differential equations under perturbation of the boundary. In the present note analogous problems are transferred to equations in partial derivatives. In problems on the perturbation of solutions under perturbation of the boundary, the first variations of the solutions were investigated in the work (^{4}) of V. A. Kronberg.

A problem with perturbed boundaries can be reduced to a problem with perturbed coefficients; for this it is sufficient to transform the perturbed domain into the unperturbed one. The coefficients of the transformed equation will depend on the transformation—this makes the calculations in the investigation of asymptotics inconvenient, and it is expedient to study directly the perturbation of the solution under perturbation of the boundary. At the same time such a replacement makes it possible to use the results of the investigation of a problem with perturbed coefficients for solving a problem with a perturbed boundary, which is used in some cases in the work.

Let Γ_ε be a family of closed curves depending on the parameter ε, whose equation in polar coordinates is given in the form $\rho = 1 + \varepsilon\alpha(\varphi)$, where $\alpha(\varphi)$ is some analytic function of φ with period 2π. Each curve Γ_ε bounds a domain Ω_ε; Γ_0 (Γ_ε for $\varepsilon = 0$) is the circle $\rho = 1$, bounding the disk Ω_0.

In Sec. 1 we consider the perturbation of solutions of a boundary-value problem; in Sec. 2, eigenvalue problems.

1. Consider in the domain Ω_ε the following perturbed problem A_ε:

$$L_\varepsilon u_\varepsilon \equiv \Delta u_\varepsilon + a(x,y)\frac{\partial u_\varepsilon}{\partial x} + b(x,y)\frac{\partial u_\varepsilon}{\partial y} + c(x,y)u_\varepsilon = f(x,y) \tag{1}$$

under perturbation of the boundary

$$u_\varepsilon\big|_{\Gamma_\varepsilon} = u_\varepsilon(1+\varepsilon\alpha(\varphi),\varphi) = 0, \tag{2}$$

where the coefficients $a(x,y)$, $b(x,y)$, $c(x,y)$ and $f(x,y)$ are analytic functions.

The unperturbed problem A_0 consists in solving the problem

$$L_0 u_0 \equiv \Delta u_0 + a\frac{\partial u_0}{\partial x} + b\frac{\partial u_0}{\partial y} + cu_0 = f(x,y), \qquad u_0\big|_{\Gamma_0} = u_0(1,\varphi) = 0.$$

Case 1, when the limiting problem is not on the spectrum. We shall seek the solution u_ε of the problem A_ε in the form

$$u_\varepsilon = u_0 + \varepsilon u_1 + \dots + \varepsilon^n u_n + \cdots \tag{3}$$

Substituting (3) into (1), (2) and comparing the coefficients of like powers of ε, we obtain the following problems:

$$L_0 u_0 \equiv \Delta u_0 + a\frac{\partial u_0}{\partial x} + b\frac{\partial u_0}{\partial y} + cu_0 = f(x,y), \qquad u_0\big|_{\Gamma_0} = u_0(1,\varphi) = 0;$$

$$L_0 u_n \equiv \Delta u_n + a\frac{\partial u_n}{\partial x} + b\frac{\partial u_n}{\partial y} + cu_n = 0, \qquad u_n\big|_{\Gamma_0} = -\sum_{l=1}^{n}\frac{\alpha^l(\varphi)}{l!}D^{(l)}u_{n-l},$$

where $D^{(l)}$ is the normal derivative of order l. Solving these problems successively, we obtain the functions $u_0, u_1, \dots, u_n, \dots$.

Passing to an arbitrary bounded domain with analytic boundary and to an arbitrary elliptic operator with analytic coefficients would lead only to a certain complication of the calculations. We note that any elliptic operator is transformed into an operator of the type considered, and any finite simply connected domain with analytic boundary can be carried by a conformal transformation into a disk; moreover, the Laplace operator does not change its form. The case of three or more variables is, in principle, no different from the case considered.

We make the change of variables

$$\bar{x} = x(1 + \varepsilon v(x, y)), \qquad \bar{y} = y(1 + \varepsilon v(x, y)),$$

$$v(x, y) = \tilde{v}(\rho, \varphi), \qquad \text{on the boundary } \ \tilde{v}(1, \varphi) = \alpha(\varphi),$$

where $v(x, y)$ is some fixed analytic function of x and y (for example, harmonic). This transformation carries the disk Ω_0 into the domain Ω_ε. By direct calculations it is not difficult to show that the Jacobian is nonzero for sufficiently small ε. Consequently, there exists a one-to-one transformation of the disk Ω_0 onto the domain Ω_ε: $(x, y) \leftrightarrow (\bar{x}, \bar{y})$, $u(x, y) = \bar{u}(\bar{x}, \bar{y})$. Under this transformation, equation (1) and boundary condition (2) are rewritten in the form

$$L_\varepsilon \bar{u} \equiv A_\varepsilon(\bar{x}, \bar{y}) \frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + B_\varepsilon(\bar{x}, \bar{y}) \frac{\partial^2 \bar{u}}{\partial \bar{x} \partial \bar{y}} + C_\varepsilon(\bar{x}, \bar{y}) \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + D_\varepsilon(\bar{x}, \bar{y}) \frac{\partial \bar{u}}{\partial \bar{x}} +$$

$$+ E_\varepsilon(\bar{x}, \bar{y}) \frac{\partial \bar{u}}{\partial \bar{y}} + g_\varepsilon(\bar{x}, \bar{y}) \bar{u} = F_\varepsilon(\bar{x}, \bar{y}), \tag{4}$$

$$\bar{u}\big|_{\Gamma_0} = \bar{u}(1, \rho) = 0, \tag{5}$$

where the coefficients are analytic functions of ε; $A_\varepsilon = 1 + O(\varepsilon)$, $B_\varepsilon = O(\varepsilon)$, $C_\varepsilon = 1 + O(\varepsilon)$.

As is known, the solution of problem (4), (5) for a differential equation of elliptic type with analytic coefficients depending analytically on ε is itself a function analytic in ε:

$$\bar{u} = u_0 + \varepsilon u_1 + \ldots + \varepsilon^n u_n + \ldots,$$

where $u_0, u_1, \ldots, u_n$ are those defined above, since the expansion of an analytic function into a power series is unique. By analyticity this series converges for sufficiently small ε, i.e., expansion (3) converges for sufficiently small ε.

Case 2, when the limiting problem lies on the spectrum. In this case we shall seek the solution u_ε of problem Λ_ε in the form

$$u_\varepsilon = \frac{c_0 u_{-1}}{\varepsilon} + (u_0 + c_1 u_{-1}) + \varepsilon(u_1 + c_2 u_{-1}) + \ldots + \varepsilon^n (u_n + c_{n+1} u_{-1}) + \ldots \tag{6}$$

Substituting (6) into (1), (2) and equating the coefficients of like powers of ε, we obtain:

$$L_0 u_{-1} \equiv \Delta u_{-1} + a\frac{\partial u_{-1}}{\partial x} + b\frac{\partial u_{-1}}{\partial y} + cu_{-1} = 0, \qquad u_{-1}\big|_{\Gamma_0} = u_{-1}(1,\varphi) = 0; \tag{7}$$

$$L_0 u_0 = f(x,y), \qquad u_0\big|_{\Gamma_0} = -c_0\alpha(\varphi)Du_{-1}; \tag{8}$$

$$L_0 u_n = 0, \qquad u_n\big|_{\Gamma_0} = -\sum_{l=0}^{n} c_{n-l}\frac{D^{(l+1)}u_{-1}}{(l+1)!}\alpha^{l+1}(\varphi) - \sum_{l=1}^{n}\frac{\alpha^l(\varphi)}{l!}D^{(l)}u_{n-l}, \tag{9}$$

where for u_n we introduce the normalization

$$\int_{(\Omega_0)} u_{-1}z_0\, d\Omega_0 = (u_{-1}, z_0) = 1, \qquad (u_n, z_0) = 0, \quad n = 0, 1, \dots, \tag{10}$$

where z_0 is the solution of the adjoint homogeneous problem

$$L_0^* z_0 \equiv \Delta z_0 - \frac{\partial}{\partial x}(az_0) - \frac{\partial}{\partial y}(bz_0) + cz_0 = 0, \qquad z_0|_{\Gamma_0} = 0,$$

$u_{-1}(x,y)$ is the eigenfunction corresponding to the simple eigenvalue $\lambda_0 = 0$ of problem (7), and the function $u_0(x,y)$ is a solution of problem (8). The solvability condition for it is as follows:

$$(f, z_0) = c_0 \int_{(\Gamma_0)} \alpha(\varphi)Du_{-1}\frac{\partial z_0}{\partial N}\, d\Gamma_0. \tag{11}$$

It is obtained from Green' s formula

$$(L_0u_0, z_0) - (u_0, L_0^*z_0) = \int_{(\Gamma_0)} \left(\frac{\partial u_0}{\partial N}z_0 - u_0\frac{\partial z_0}{\partial N}\right) d\Gamma_0 +$$

$$+ \int_{(\Gamma_0)} [a\cos(N,x) + b\cos(N,y)]u_0z_0\, d\Gamma_0.$$

Condition (11) can be satisfied for any $f(x,y)$, by choosing c_0, if and only if

$$\int_{(\Gamma_0)} \alpha(\varphi)Du_{-1}\frac{\partial z_0}{\partial N}\, d\Gamma_0 \neq 0. \tag{12}$$

If condition (12) is fulfilled, then we determine c_0; knowing c_0, we find u_0 as a particular solution of problem (8) under the normalization condition.

Suppose that all $c_0, c_1, \dots, c_{n-1}, u_0, u_1, \dots, u_{n-1}$ have already been determined. From the solvability condition for problem (9)

$$0 = -\int_{(\Gamma_0)} u_n \frac{\partial z_0}{\partial N}\, d\Gamma_0 = \int_{(\Gamma_0)} \left[\sum_{l=0}^{n} c_{n-l} \frac{\alpha^{l+1}(\varphi)}{(l+1)!} D^{(l+1)} u_{-1} + \sum_{l=1}^{n} \frac{\alpha^{l}(\varphi)}{l!} D^{(l)} u_{n-l} \right] \frac{\partial z_0}{\partial N}\, d\Gamma_0$$

we find c_n, and then from problem (9) determine a particular solution u_n under condition (10).

Let us note that the boundary-value problem with a perturbed boundary can be reduced to a problem with perturbed (analytic) coefficients. By virtue of paper (1), after the transformation there is an expansion (6), which is analytic for $\varepsilon \neq 0$.

In the case where the problem A_ε has associated functions, we shall seek the solution u_ε in the form

$$u_\varepsilon = \frac{c_0 \bar{u}_0}{\varepsilon^k} + \frac{c_0 \bar{u}_1 + c_1 \bar{u}_0}{\varepsilon^{k-1}} + \dots + \frac{c_0 \bar{u}_{k-1} + \dots + c_{k-1} \bar{u}_0}{\varepsilon} +$$

$$+ \left[u_0 + (c_1 \bar{u}_{k-1} + \dots + c_k \bar{u}_0) \right] + \dots$$

$$\dots + \varepsilon^s \left[u_s + (c_{s+1} \bar{u}_{k-1} + \dots + c_{k+s} \bar{u}_0) \right] + \dots,$$

where $c_0, c_1, \dots, c_k, c_{k+1}, \dots, c_{k+s}$, $\bar{u}_0, \bar{u}_1, \dots, \bar{u}_{k-1}$, $u_0, u_1, \dots, u_s$ are determined in the same way as above; the functions $\bar{u}_1, \bar{u}_2, \dots, \bar{u}_{k-1}$ are called associated functions of the eigenfunction $\bar{u}_0$ (see (1)).

2. We shall consider in the domain Ω_ε the perturbed eigenvalue problem

$$L_\varepsilon u_\varepsilon - \lambda_\varepsilon u_\varepsilon \equiv \Delta u_\varepsilon + a(x,y) \frac{\partial u_\varepsilon}{\partial x} + b(x,y) \frac{\partial u_\varepsilon}{\partial y} + c(x,y) u_\varepsilon - \lambda_\varepsilon u_\varepsilon = 0 \qquad (13)$$

under perturbation of the boundary

$$u_\varepsilon \big|_{\Gamma_\varepsilon} = u_\varepsilon(1 + \varepsilon \alpha(\varphi), \varphi) = 0, \qquad (14)$$

where the coefficients $a(x,y)$, $b(x,y)$, and $c(x,y)$ are analytic functions.

For $\varepsilon = 0$ one obtains the unperturbed problem:

$$L_0u_0-\lambda_0u_0\equiv\Delta u_0+a\frac{\partial u_0}{\partial x}+b\frac{\partial u_0}{\partial y}+cu_0-\lambda_0u_0=0,\qquad u_0\big|_{\Gamma_0}=u_0(1,\varphi)=0.$$

Here u_0 is the normalized eigenfunction; for the time being we assume the eigenvalue λ_0 to be a simple eigenvalue. We seek the eigenvalue λ_ε close to λ_0 and the corresponding eigenfunction u_ε for the perturbed operator in the form

$$\lambda_\varepsilon=\sum_{n=0}^{\infty}\varepsilon^n\lambda_n,\qquad u_\varepsilon=\sum_{n=0}^{\infty}u_n\varepsilon^n,\qquad u_0\neq 0. \tag{15}$$

Substituting (15) into (13), (14) and comparing coefficients of like powers of ε, we obtain

$$L_0u_0-\lambda_0u_0=0,\qquad u_0\big|_{\Gamma_0}=u_0(1,\varphi)=0; \tag{16}$$

$$L_0u_n-\lambda_0u_n=\sum_{k=1}^{n}\lambda_ku_{n-k},\qquad u_n\big|_{\Gamma_0}=-\sum_{l=1}^{n}\frac{\alpha^l(\varphi)}{l!}D^{(l)}u_{n-l}, \tag{17}$$

where $D^{(l)}$ is the derivative in the normal direction of order l; moreover, for u_n we introduce the normalization

$$\int_{(\Omega_0)}u_0z_0\,d\Omega_0=(u_0,z_0)=1,\qquad (u_n,z_0)=0,\qquad n\neq 0. \tag{18}$$

To determine u_1 and λ_1 we consider the problem

$$L_0u_1-\lambda_0u_1=\lambda_1u_0,\qquad u_1\big|_{\Gamma_0}=-\alpha(\varphi)Du_0. \tag{19}$$

The condition for its solvability is

$$\lambda_1=\int_{(\Gamma_0)}\alpha(\varphi)Du_0\cdot\frac{\partial z_0}{\partial N}\,d\Gamma_0.$$

It can be obtained from Green' s formula, using the equations, the boundary conditions, and (18).

From the solvability condition we find λ_1 and then determine u_1 as a particular solution of problem (19) subject to the normalization condition.

Suppose that all $\lambda_0,\lambda_1,\dots,\lambda_{n-1}$ and $u_0,u_1,\dots,u_{n-1}$ have already been determined; to determine λ_n and u_n we consider the problem

$$L_0 u_n - \lambda_0 u_n = \sum_{k=1}^{n} \lambda_k u_{n-k}, \qquad u_n\big|_{\Gamma_0} = -\sum_{l=1}^{n} \frac{\alpha^l(\varphi)}{l!} D^{(l)} u_{n-l}. \qquad (20)$$

From the solvability condition for problem (20) we find the eigenvalue λ_n, and then determine u_n as a particular solution of problem (20).

In a manner analogous to that indicated above, the perturbed problem with perturbed boundary can be reduced to a problem with perturbed (analytic) coefficients. In doing so we obtain that the expansions of the eigenvalue λ_ε and the eigenfunction u_ε are analytic for sufficiently small $\varepsilon \neq 0$.

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Note: Figure translations are in progress. See original paper for figures.