Reports of the Academy of Sciences of the USSR

MATHEMATICS

1964

SovietRxiv

Abstract

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Reports of the Academy of Sciences of the USSR
1964, Volume 158, No. 2

MATHEMATICS

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ON BOUNDARY-VALUE PROBLEMS FOR PARTIAL DIFFERENTIAL EQUATIONS WITH A DIFFERENTIAL BESSEL OPERATOR

(Presented by Academician I. N. Vekua on March 20, 1964)

In the theory of boundary-value problems for elliptic differential equations, a priori estimates of the norms of solutions of boundary-value problems in terms of the norms of the right-hand sides of the differential equation and the norms of the boundary conditions play a major role. Many interesting works are devoted to a priori estimates of solutions of elliptic equations (see, for example, ([3–5])).

In the present note, in the metrics of the corresponding weighted classes, a priori estimates are given for the norms of solutions of boundary-value problems for partial differential equations with a differential Bessel operator. For simplicity of exposition we restrict ourselves to the consideration of the half-space $x \geq 0$ in the two-dimensional Euclidean space R_2.

1. Consider the set K_x of all functions $f(x, y)$, each of which is infinitely differentiable, even in x, and finite with respect to the set of variables x, y. On this set we define the differential operator

$$\mathcal{D}_x f = \frac{\partial f(x,y)}{x\,\partial x} \tag{1}$$

and powers of this operator

$$\mathcal{D}_x^k f = \mathcal{D}_x(\mathcal{D}_x^{k-1} f). \tag{2}$$

The functional space $W^{(l)}_{x,2,\gamma}(\overset{+}{R}_2)$, where l is a nonnegative number, is defined as the closure of the set K_k of functions $f(x, y)$ with respect to the norm

$$\|f\|^2_{W^{(l)}_{x,2,\gamma}(\overset{+}{R}_2)}.$$

The definitions of the norms $\|f\|_{W^{(l)}_{x,2,\gamma}(\overset{+}{R}_2)}$ and $\|f\|_{W^{(l_1)}_{y,2,\gamma}(\overset{+}{R}_2)}$ are given in work ([7]).

The functional space $W^{(l,l_1)}_{x,y,2,\gamma}(\overset{+}{R}_2)$ is defined as the closure of the set K_x with respect to the norm

$$\|f\|^2_{W^{(l,l_1)}_{x,y,2,\gamma}(\overset{+}{R}_2)} = \|f\|^2_{W^{(l)}_{x,2,\gamma}(\overset{+}{R}_2)} + \|f\|^2_{W^{(l_1)}_{y,2,\gamma}(\overset{+}{R}_2)}. \tag{3}$$

Here the index l takes not only positive integer values. The case $\gamma = 0$ is contained in note ([1]).

Let $\{f_i\}$ be a sequence of functions belonging to K_x, converging in the norm $W^{(l,l_1)}_{x,y,2,\gamma}(\overset{+}{R}_2)$ to the function $f(x)$. We shall say that the function

$$\varphi^{(k)} = \left.\frac{\partial^k f}{(x,\partial x)^k}\right|_{x=0} \qquad (k \geq 0)$$

is the trace of the derivative $\partial^k f/(x\partial x)^k$ at $x = 0$, if, as $n \to \infty$,

$$\left\| \varphi^{(k)} - \left.\frac{\partial^k f_n}{(x,\partial x)^k}\right|_{x=0} \right\|_{L_{2,\gamma}(R_1)} \to 0.$$

The following theorems on traces hold.

Theorem 1. Let $f \in W^{(l,l_1)}_{x,y,2,\gamma}(\overset{+}{R}_2)$, and suppose that for nonnegative integers k the inequality

$$\mu = \mu(k) = 1 - \frac{2k}{l} - \frac{2\gamma+1}{2l} > 0 \tag{4}$$

is satisfied. Then there exist traces of the derivatives $\partial^k f(x,y)/(x\partial x)^k$ on $x = 0$, and they belong to $W^{(\bar{l}_1)}_{y,2}(R_1)$, $\bar{l}_1 = \mu l_1$. Moreover,

$$\left\| \left.\frac{\partial^k f}{(x\partial x)^k}\right|_{x=0} \right\|_{W^{(\bar{l}_1)}_{y,2}(R_1)} \leqslant c\|f\|_{W^{(l,l_1)}_{x,y,2,\gamma}(\overset{+}{R}_2)}, \tag{5}$$

where the constant c does not depend on f.

Remark. Here and below $W^{(\bar{l}_1)}_{y,2}(R_1)$ denotes the usual Aronszajn-Slobodetskii space (see ([2])).

The converse assertion also holds.

Theorem 2. Let nonnegative integers k be given for which the inequality

$$\mu = \mu(k) = 1 - \frac{2k}{l} - \frac{2\gamma + 1}{2l} > 0 \tag{6}$$

is satisfied. For such k, prescribe on R_1 functions $\varphi^{(k)}(y) \in W_{y,2}^{(\bar{l}_1)}(R_1)$, with $\bar{l}_2 = \mu l_1$.

There exists a function $\bar{f} \in W_{x,y,2,\gamma}^{(l,l_1)}(\overset{+}{R}_2)$ such that the $\varphi^{(k)}$ are the traces of the corresponding $\partial^k \bar{f}/(x\partial x)^k$ on $x = 0$, and

$$\lim_{x \to 0} \left\| \frac{\partial^k \bar{f}}{(x\partial x)^k} - \varphi^{(k)}(y) \right\|_{W_{x,2}^{(\bar{l}_1)}(R_1)} = 0 \tag{7}$$

for all admissible k. Moreover,

$$\|\bar{f}\|_{W_{x,y,2,\gamma}^{(l,l_1)}(\overset{+}{R}_2)} \leqslant c \sum_k \|\varphi^{(k)}(y)\|_{W_{y,2}^{(\bar{l}_1)}(R_1)} \tag{8}$$

with a constant c independent of $\varphi^{(k)}$.

In constructing the space $W_{x,2,\gamma}^{(l)}(\overset{+}{R}_2)$ in the norm defined by formula (3), the differential operators of the form

$$x^k \frac{\partial^k}{(x\partial x)^k} \qquad (k = 1, 2, \dots, l) \tag{9}$$

are involved.

In the case when the number k is even and $\gamma > 0$, the indicated differential operators in the norm (3) may be replaced by the corresponding powers of the Bessel operator

$$\mathfrak{B}_x = \frac{\partial^2}{\partial x^2} + \frac{2\gamma}{x}\frac{\partial}{\partial x} \qquad (\gamma > 0). \tag{10}$$

The norm thereby obtained will be equivalent to the former norm (3).

In exactly the same way, the trace theorems stated above can be formulated not only in terms of the operators $\partial^k/(x\partial x)^k$, but also in terms of powers of the Bessel operator.

Let $R_2^{(+,+)}$ denote the domain $x \geqslant 0$, $y \geqslant 0$ of the two-dimensional Euclidean space R_2. Define, similarly to the preceding, the functional space $W_{x,y,2,\gamma}^{(l,l_1)}(R_2^{(+,+)})$.

Theorem 3. Let $f \in W^{(l,l_1)}_{x,y,z,\gamma}(R_2^{(+,+)})$, and suppose that for integer nonnegative numbers r the inequality

$$\mu = \mu(r) = 1 - \frac{r}{l} - \frac{1}{2l} > 0$$

is satisfied.

Then the traces of the derivatives $\partial^r f/\partial x^r$ on $x = 0$ exist and belong to

$$W^{(\bar{l}_1)}_{y,z,\gamma}(R_1)$$

with $\bar{l}_1 = \mu l_1$. Moreover,

$$\left\| \frac{\partial^r f}{\partial x^r}\bigg|_{x=0} \right\|_{W^{(\bar{l}_1)}_{y,z,\gamma}(R_1)} \le c\|f\|_{W^{(l,l_1)}_{x,y,z,\gamma}(R_2^{(+,+)})},$$

where the constant c does not depend on f.

As in the preceding case, the converse theorem is also true.

2. Let $\overset{+}{R}_2$ denote the half-plane $x > 0$ of the two-dimensional Euclidean space R_2 of points $z = (x, y)$. Let

$$\mathcal{L} = \mathcal{L}\left(z; \mathfrak{B}_x, \frac{\partial}{\partial y}\right) = \sum_{i=0}^{k} A^{(i,k)}(z)\mathfrak{B}_x^{k-i}\frac{\partial^{2i}}{\partial y^{2i}} + \sum_{\tau+2\nu\le 2k-1} C^{(\tau,\nu)}(z)\mathfrak{B}_x^{\nu}\frac{\partial^\tau u}{\partial y^\tau} + D(z) \tag{11}$$

be a linear differential operator of order $2k$, whose coefficients $A^{(i,k)}(z)$, $C^{(\tau,\nu)}(z)$, and $D(z)$ are real functions of z, defined in the closed domain $\bar{\Omega}$ ($\overset{+}{R}_2 = \Omega$). Here, as above, $\mathfrak{B}_x$ denotes the Bessel operator

$$\frac{\partial^2}{\partial x^2} + \frac{2\gamma}{x}\frac{\partial}{\partial x} \qquad (\gamma > 0). \tag{12}$$

We shall say that the operator $\mathcal{L}$ is B-elliptic in Ω, if for every $z \in \bar{\Omega}$ and every real vector $\alpha = (\alpha_1, \alpha_2)$ $(\alpha_1 \ge 0)$ the inequality

$$\left|\mathcal{L}_0(z; (i\alpha_1)^2, i\alpha_2)\right| \ge \delta|\alpha|^{2k}, \tag{13}$$

holds, where δ is a positive number. The expression $\mathcal{L}_0(z; (i\alpha_1)^2, i\alpha_2)$ is obtained from $\mathcal{L}_0\left(z; \mathfrak{B}_x, \partial/\partial y\right)$ by replacing the symbol of the Bessel operator by the number $(i\alpha_1)^2$, and the symbol of differentiation $\partial/\partial y$ by the number $i\alpha_2$. The operator

$$\mathcal{L}_0 = \mathcal{L}_0\left(z; \mathfrak{B}_x, \frac{\partial}{\partial y}\right)$$

has the form

$$\mathcal{L}_0 = \mathcal{L}_0\left(z; \mathfrak{B}_x, \frac{\partial}{\partial y}\right) = \sum_{i=0}^{k} A^{(i,k)}(z)\mathfrak{B}_x^{k-i}\frac{\partial^{2i}}{\partial y^{2i}} \tag{14}$$

—the principal part of the operator $\mathcal{L}$.

In what follows we shall consider the functional space

$$W^{(l,l_1)}_{x,y,z,\gamma}(\overset{+}{R}_2)$$

in the case $l = l_1$. Such spaces will be denoted by

$$W^{(l)}_{2,\gamma}(\overset{+}{R}_2).$$

Here $\overset{+}{R}_2$ will be understood to mean the closed domain.

Theorem 4. Let $\mathcal{L}$ be a linear B-elliptic differential operator with constant coefficients, containing only differential operators of order $2k$. Let the natural number $l \geq 2k$, and let

$$f \in W^{(l-2k)}_{2,\gamma}(\overset{+}{R}_2).$$

Then the equation $\mathcal{L}u + u = f$ has a unique solution

$$u \in W^{(l)}_{2,\gamma}(\overset{+}{R}_2),$$

for which the inequality

$$\|u\|_{W^{(l)}_{2,\gamma}(\overset{+}{R}_2)} \leq c\|f\|_{W^{(l-2k)}_{2,\gamma}(\overset{+}{R}_2)} \tag{15}$$

holds, where c does not depend on f.

Theorem 5. Let $\mathcal{L}$ be a linear B-elliptic operator of order $2k$, defined in Ω. Suppose that the coefficients of the operator have continuous and bounded corresponding derivatives up to order $l-2k$, the leading coefficients are continuous in Ω, and let $l \geq 2k$. Then for every $u \in W^{(l)}_{2,\gamma}(\mathbb{R}^+_2)$ the inequality

$$\|u\|_{W^{(l)}_{2,\gamma}(\mathbb{R}^+_2)} \leq c\left[\|\mathcal{L}u\|_{W^{(l-2k)}_{2,\gamma}(\mathbb{R}^+_2)} + \|u\|_{L_{2,\gamma}(\mathbb{R}^+_2)}\right], \tag{16}$$

holds, where c does not depend on u.

Theorem 6. Let $\mathcal{L}$ be a linear B-elliptic operator of order $2k$, defined in $R_2^{(+,+)}$, and let R_μ be differential operators of orders m_μ ($\mu = 1, 2, \dots, k$), defined on $x = 0$ (on the part of the boundary of the domain $R_2^{(+,+)}$); moreover, let the operators $\mathcal{L}$ and R_μ satisfy the Lopatinskii condition on $x = 0$. Suppose $l \geq 2k$ and the numbers $m_\mu \leq l-1$. Suppose that the coefficients of $\mathcal{L}$ have continuous and bounded corresponding derivatives up to order $l - 2k$, and the coefficients of R_μ have continuous and bounded derivatives up to the corresponding orders. The leading coefficients of $\mathcal{L}$ and R_μ are continuous in the corresponding closed domains. Then for every function $u \in W_{2,\gamma}^{(l)}(R_2^{(+,+)})$ the inequality

$$\|u\|_{W_{2,\gamma}^{(l)}(R_2^{(+,+)})} \leq c \left[\|\mathcal{L}u\|_{W_{2,\gamma}^{(l-2k)}(R_2^{(+,+)})} + \sum_{\mu=1}^{k} \|R_\mu u\|_{W_{2,\gamma}^{(l-m_\mu-1/2)}(S_2)} + \|u\|_{L_{2,\gamma}(R_2^{(+,+)})} \right], \tag{17}$$

where c does not depend on u, and S_2 denotes the part of the boundary of the domain $R_2^{(+,+)}$ corresponding to $x = 0$.

Represent R_n in the form $R_n = R_m \times R_{n-m}$. Introduce in R_{n-m} spherical coordinates with origin at the point $x_0^{(n-m)}$. Then any point $x = (x_1, \dots, x_n) \in R_n$ is represented in the form $x = (x^{(m)}, x_0^{(n-m)}, \rho, \omega_{n-m})$. Let

$$\mathcal{L} = \mathcal{L}\left(x; \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}, \mathcal{B}_\rho\right) = \sum_{2j+r \leq 2k} \sum_{i_1, \dots, i_r = 1}^{m} A^{(i_1, \dots, i_r, j)}(x) \frac{\partial^r}{\partial x_{i_1} \cdots \partial x_{i_r}} \mathcal{B}_\rho^j \tag{18}$$

be a linear differential operator of order $2k$, whose coefficients are real functions defined in R_n. Here $\mathcal{B}_\rho$ denotes the Bessel operator

$$\frac{\partial^2}{\partial \rho^2} + \frac{2\gamma}{\rho} \frac{\partial}{\partial \rho} \qquad (\gamma > 0).$$

We shall call the operator $\mathcal{L}$ B-elliptic in R_n if, for every $x \in R_n$ and every real vector $\alpha = (\alpha_1, \dots, \alpha_m, \alpha_{m+1})$, the inequality

$$\left|\mathcal{L}_0\left(x; i\alpha_1, \dots, i\alpha_m, (i\alpha_{m+1})^2\right)\right| \geq \delta |\alpha|^{2k},$$

holds, where δ is a positive number and $\mathcal{L}_0$ is the principal part of the operator $\mathcal{L}$. For this operator, theorems of the type of Theorems 4 and 5 are also valid in terms of the space $W_{x^{(m)}, \rho, 2, \gamma}^{l^{(m)}, l}(R_n)$. We note that in other similar spaces one can also indicate B-elliptic operators with the corresponding coercivity inequalities. In accordance with what was indicated above, B-parabolic operators can also be considered.

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Received
16 III 1964

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Note: Figure translations are in progress. See original paper for figures.