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A. A. BANDZAITIS, A. V. KAROSIENE, A. Yu. SAVUKINAS,

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Abstract

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A. A. BANDZAITIS, A. V. KAROSIENE, A. Yu. SAVUKINAS,
Academician of the Academy of Sciences of the Lithuanian SSR A. P. YUTSIS

QUANTITIES OF THE THEORY OF ANGULAR MOMENTUM WITH NEGATIVE PARAMETERS REPRESENTING QUANTUM NUMBERS OF ANGULAR MOMENTUM

The eigenvalue equation

$$\mathbf{j}^2\psi(jm) = j(j+1)\psi(jm), \tag{1}$$

where $\mathbf{j}^2$ is the operator of the square of the angular momentum, is not changed if the quantum number j is replaced in the following way:

$$j \to \bar{j} = -j - 1. \tag{2}$$

This corresponds to the classical case, when the square of a real number is not changed when the sign of the base is changed.

In the mathematical apparatus of angular momentum, as a rule, positive values of the quantum number of angular momentum are used. However, it proves useful to extend this apparatus to the case of negative values of j. The present note is devoted to this question.

It is obvious that under substitution (2) in the wave function only the phase factor can change. It turns out that, in order to obtain a consistent system of phases adjoining the standard system of phases ([1]), the following assumption is necessary:

$$\psi(\bar{j}m) = (-1)^{j-m}\psi(jm). \tag{3}$$

This means that if the function $\psi(jm)$ is transformed according to the representation of rotations of three-dimensional space $D^{(j)}$, then $\psi(\bar{j}m)$ is transformed according to the representation

$$D^{(\bar{j})} = UD^{(j)}U^{-1}, \tag{4}$$

where U is a unitary matrix with elements

$$u_{mm'} = (-1)^{j-m}\delta(m, m'). \tag{5}$$

In the mathematical apparatus of vector addition of angular momenta, the Clebsch–Gordan coefficients play an especially important role. They are expressed as single sums of quantities consisting of factorials of linear combinations of the parameters of these coefficients. Substitution (2) entails the consequence that some of these linear combinations of parameters become negative. The formulas for the Clebsch–Gordan coefficients possess the interesting property that, under substitution (2), the number of factorials of negative quantities is the same in the numerator and in the denominator. Therefore the relation

$$\frac{(-a)!}{(-b)!} = \frac{(-1)^{b-1}(b-1)!}{(-1)^{a-1}(a-1)!} = (-1)^{b-a}\frac{(b-1)!}{(a-1)!}, \tag{6}$$

obtained by finding the limit of the ratio between two Gauss Π-functions when they approach their poles, can be successfully applied. The exponents $a-1$ and $b-1$ denote the numbers of negative factors. With them only such operations may be performed as with exponents of powers with any base. Thus, one cannot write $(-1)^{a-b}$ instead of $(-1)^{b-a}$, since

the whole quantity (6) may be under the square root, which is essential in the proposed mathematical apparatus. As a consequence, when substitution (2) is made, each negative factor under the square root must be regarded as an imaginary unit multiplied by the square root of the absolute value of this factor. Thus, for example, on the right-hand side of the equation

$$(j_x \pm i j_y)\psi(jm) = \sqrt{(j \mp m)(j \pm m + 1)}\,\psi(jm \pm 1) \tag{7}$$

the square root, under substitution (2), will go over into the same root with a minus sign, which compensates the phase factor from (3). This means that (7) is also an invariant relation with respect to substitution (2) under condition (3), the phase factor of the latter condition being caused by the necessity of invariance of equation (7).

When substitution (2) is made for individual parameters representing quantum numbers of angular momentum, the formulas for the Clebsch–Gordan coefficients (cf. (1)) pass into one another or into themselves up to a phase factor. In addition, other types of formulas are obtained, which have not been used until now. However, in such a case an ambiguity up to a factor ± 1 appears, which makes these formulas inconvenient for the present problem. To avoid the appearance of new formulas, one should apply the corresponding symmetry properties and only then carry out substitution (2).

In the indicated way we obtain

$$\begin{bmatrix} \bar{j}_1 & \bar{j}_2 & \bar{j} \\ m_1 & m_2 & m \end{bmatrix} = (-1)^{j_1+j_2-j} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}. \tag{8}$$

However, from the practical point of view, the cases in which substitution (2) is applied not to all three parameters are more important. The most important are the Clebsch–Gordan coefficients with two negative parameters representing quantum numbers of angular momentum. Of these we give the following case:

$$\begin{bmatrix} \bar{j}_1 & j_2 & \bar{j} \\ m_1 & m_2 & m \end{bmatrix} = (-1)^{j_2+m_2} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}. \tag{9}$$

This relation is convenient for calculating Clebsch–Gordan coefficients for a given value of j_2. Then we have

$$\begin{bmatrix} j_1 & j_2 & j_1+k \\ m_1 & m_2 & m_1+m_2 \end{bmatrix} = (-1)^{j_2+m_2} \begin{bmatrix} \bar{j}_1 & j_2 & \bar{j}_1-k \\ m_1 & m_2 & m_1+m_2 \end{bmatrix}, \tag{10}$$

where $j_2 \geq k \geq -j_2$. (10) shows that the case $j = j_1 + k$ can be obtained from the case $j = j_1 - k$ by the substitution $j_1 \to \bar{j}_1$. This, incidentally, makes it possible to reduce the tables of formulas ([3]) for the Clebsch–Gordan coefficients almost by half.

In a similar way, for the reduced matrix (submatrix) element of the spherical function ([4]) we obtain

$$(l\|C^{(\bar{l}'+k)}\|l') = i(-1)^l (l\|C^{(l'-k)}\|l'), \tag{11}$$

where $l' \geq k \geq -l'$.

The next important quantity in the mathematical apparatus of the theory of angular momentum is the $6j$-coefficient (cf. ([4])), which is also expressed by a single sum of fractions consisting of factorials. Therefore formula (6) is applicable here as well. We give the following important cases:

$$\begin{Bmatrix} \bar{a} & b & e \\ d & c & f \end{Bmatrix} = (-1)^{b+f-e-c+1} \begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}, \tag{12}$$

$$\begin{Bmatrix} \bar{a} & \bar{b} & e \\ d & c & f \end{Bmatrix} = i(-1)^{c+d+e+2f} \begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}. \tag{13}$$

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.